Distributed Optimization of Cascade Pumping Stations Based on State-based Potential Game

Yikang Zhao, Gang Xie, Rong Li, Xinying Xu

Abstract—**To solve the centralized optimization scheduling problem of long-distance water diversion in cascade pumping stations, we propose a fully distributed optimization method based on the state-based potential game (SPG). The primary objective is to minimize the daily electricity operating cost for the cascade pumping stations. Handling the coupled equality constraints of head and water diversion volume is challenging. Inspired by the Lagrangian approach, our SPG framework uses states and state transition functions to manage these constraints effectively. This method uses graph theory and distributed optimization algorithms to achieve stationary Nash equilibrium without requiring central control. The results demonstrate that using a distributed optimization method based on the state-based potential game is feasible and efficient for solving the optimization scheduling problem of cascade pumping stations.**

Index Terms—**Cascade pumping stations, State-based potential game, Distributed optimization, Stationary Nash equilibrium**

I. INTRODUCTION

IN recent decades, climate change and pollution have exacerbated drought conditions. Population growth and poor water management have increased the demand for scarce freshwater resources [1], [2]. Water resources are unevenly distributed spatially worldwide [3]. Effective water management and allocation are crucial to ensure reliable access to adequate and safe water [4]. Cascade pumping stations are key in long-distance water diversion projects such as the South-to-North Water Diversion Project [5] and the Wanjiazhai Yellow River Diversion Project [6]. However, these long-distance water diversion projects consume a lot of electricity and contribute to high carbon emissions [7], [8]. Therefore, optimizing the daily operation of cascade pumping stations is crucial to ensure the efficient supply of water resources, reduce operating costs, and lessen the environmental impact.

Currently, the optimal scheduling problem of cascade pumping stations is mainly studied using centralized optimization methods [9], which include deterministic and stochastic optimization algorithms [10]. Deterministic algorithms such as Linear Programming (LP), Nonlinear Programming (NLP), Integer Programming (IP), and Dynamic Programming (DP) are predictable but are prone to get trapped in local optimum, especially for complex or unstructured problems [11]. Stochastic algorithms such as Genetic Algorithms (GA) [12], [13], Particle Swarm Optimization (PSO) [14], [15], and Ant Colony Optimization (ACO) [16] have strong global search capabilities but suffer from slow convergence and unstable results [17], [18], [19].

Centralized optimization algorithms necessitate bidirectional communication between the central control system and each node [20]. This requirement can lead to several limitations, including high communication demands, significant computational burden, and low flexibility and scalability [21]. In contrast, distributed optimization algorithms eliminate the need for a central control system [22]. A certain spatial distance for long-distance water diversion projects separates each single-stage pumping station. Traditional methods rely heavily on the central control system [23], which limits the full utilization of the performance capabilities of each single-stage pumping station. Therefore, we adopt a distributed optimization algorithm to address the optimal scheduling problem of cascade pumping stations.

Many studies have been on distributed optimization algorithms so far [24]. In this paper, we focus on the distributed first-order discrete-time algorithms [25]. For cascade pumping stations, there are antagonistic or cooperative relationships between different levels of pumping stations [26], [27], [28]. Therefore, game models such as Stackelberg game [29], [30], cooperative game [31], and potential game[32] are introduced to describe these relationships [33]. Among these, the potential game has a potential function that uniformly describes the changes in all agents [34]. Each agent converges to a Nash equilibrium. An auxiliary state space can be introduced to handle complex centralized targets and coupled constraints, and the game framework is extended to a state-based potential game (SPG) [35].

Distributed optimization methods aim to minimize the daily electricity operating cost [36]. However, they often overlook the competition or cooperation between each single-stage pumping station [37], [38]. Designing a potential function ensures that strategy updates of all single-stage pumping stations move toward the global optimum. SPG carefully analyzes information interactions and state transmission among single-stage pumping stations by introducing state variables. The properties of the potential function guarantee system convergence and stability.

This paper proposes a state-based potential game (SPG) to solve the distributed optimization problem of cascade pumping stations with coupled equality constraints. The main contributions are as follows:

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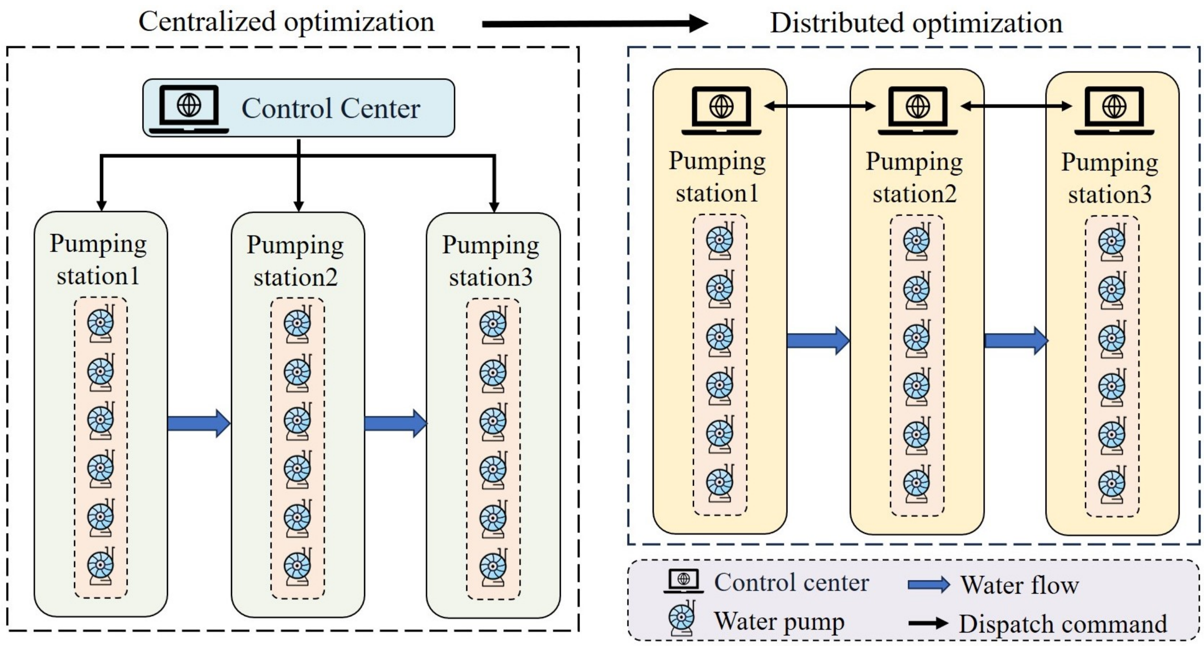

Fig. 1. Changes in optimization scheduling method

(1)The fully distributed optimization framework of cascade pumping stations is designed based on the state-based potential game. Each pumping station acts as an independent game agent and locally optimizes its strategies when solving the coupled constraint problem.

(2)Cascade pumping stations update strategies based solely on local information and estimates from neighboring stations about coupling constraints. This approach eliminates the need to know the specific parameters of other pumping stations. It greatly reduces communication requirements and preserves each user's privacy.

(3)The designed optimization algorithm has good convergence performance. It ensures that optimization strategies are adjusted according to the Time-of-Use (TOU) electricity pricing. It can also converge to the stationary Nash equilibrium for different water diversion volumes and network topologies.

II. CONSTRUCTION OF OPERATION MODEL FOR CASCADE PUMPING STATIONS

A. Problem Description

In this section, we study the water pump systems, which include cascade pumping stations, single-stage pumping stations, and individual water pumps. Cascade pumping stations are composed of multiple single-stage pumping stations connected in series to extend water diversion distances. Single-stage pumping stations consist of multiple water pumps connected in parallel or series to increase water diversion volume.

In centralized optimization, a control center regulates all equipment in the system. Each single-stage pumping station has a computer monitoring system to execute commands. However, we aim to enable each single-stage pumping station to act as an independent agent. These pumping stations will control their internal water pumps and communicate with adjacent pumping stations. This approach removes the influence of a centralized control center and achieves fully distributed optimization. For example, Fig. 1 shows a system with three single-stage pumping stations. Each station is equipped with six water pumps.

B. Centralized Optimization Model

Electricity costs are the most critical factor in operating cascade pumping stations. Therefore, we construct an optimization scheduling model to minimize the operating cost. The model considers N-level single-stage pumping stations within cascade pumping stations. Each single-stage pumping station contains M water pumps. The daily water diversion task is divided into 7 time periods according to Time-of-Use (TOU) electricity pricing.

1) Objective Function: The minimum objective function for cascade pumping stations is as follows.

$$F\text{=min}\sum_{i=1}^{N}\sum_{k=1}^{T}\sum_{m=1}^{M}\frac{\rho\cdot g\cdot Q\cdot H_i^k\cdot t_{i,m}^k}{1000\cdot\eta_{im}\cdot\eta_{mot}\cdot\eta_{dr}\cdot\eta_f}\cdot p^k \quad (1)$$

where F denotes the daily electricity operating cost for cascade pumping stations, N represents the number of single-stage pumping stations, T represents the total number of periods, which is divided according to TOU electricity pricing during the day, M represents the number of water pumps in a single-stage pumping station, ρ is the density of the liquid, g is the acceleration of gravity, Q is the flow rate of the water pump, H_i^k is the head of i^{th} single-stage pumping station in the k^{th} period, $t_{i,m}^k$ is running time of the m^{th} water pump in the k^{th} period, η_{im} is the efficiency of water pump, η_{mot} is the motor efficiency, η_{dr} is the unit drive efficiency, η_f is the efficiency of the frequency converter, p_k denotes TOU electricity price in the k^{th} period.

2) Constraints: There are water diversion distances between the single-stage pumping stations. In the distributed optimization, Each pumping station acts as a separate agent. If the water loss between each pumping station is not considered, the following (2) must be satisfied.

$$\sum_{k=1}^{T}\sum_{m=1}^{M}\left(Q\cdot t_{i,m}^{k}\cdot 3600\right)=Q_{\mathrm{d}} \tag{2}$$

where Q_{d} represents the total water volume.

In addition, there are coupled equality constraints about the head of each single-stage pumping station.

$$\sum_{i=1}^{N}H_{i}^{k}=Z_{out}^{k}-Z_{in}^{k}+\Delta\mathrm{h} \tag{3}$$

To ensure the operational efficiency and safety performance of the entire system, equation (3) is set as the overall hydraulic head balance constraint of the cascade pumping stations, where Δh are hydraulic losses, Z_{out}^{k} is the water level of the last afterbay of cascade pumping stations in the k^{th} period, Z_{in}^{k} is the water level of the first forebay of cascade pumping stations in the k^{th} period.

Moreover, running time and head must satisfy upper and lower limit constraints. During operation, because of actual hydraulic losses, the head of each single-stage pumping station should be lower than the water level difference between the forebay and afterbay.

$$t_{\min}^{k}\le t_{i,m}^{k}\le t_{\max}^{k} \tag{4}$$
$$H_{\min}^{k}\le H_{i}^{k}\le H_{\max}^{k} \tag{5}$$

The water levels of the forebay and afterbay reflect the actual operation of the single-stage pumping station.

$$Z_{i,in,\min}^{k}\le Z_{i,in}^{k}\le Z_{i,in,\max}^{k} \tag{6}$$
$$Z_{i,out,\min}^{k}\le Z_{i,out}^{k}\le Z_{i,out,\max}^{k} \tag{7}$$

where $Z_{i,in}^{k}$ is water level of the forebay of i^{th} single-stage pumping station, $Z_{i,out}^{k}$ is water level of the afterbay of i^{th} single-stage pumping station.

III. STATE-BASED POTENTIAL GAME OF CASCADE PUMPING STATIONS

Under two-way information interaction, we can construct a state-based potential game within a distributed framework. The framework consists of two main parts: the design of game rules and the development of distributed optimization algorithms, which can be developed independently.

The design of game rules includes four key aspects:

(1)The game model maps single-stage pumping stations as agents.

(2)The strategy space in the optimization problem is mapped to the state space and actions in the game model.

(3)Local and global optimization objectives are mapped to the agent cost and potential functions.

(4)Considering the impact of information interaction during optimization, the optimization iteration process is treated as repeated game stages in the game model.

A. Design of Game Rules

The state-based potential game integrates the state of the underlying space into the game-theoretic environment. It is defined as $G=\{N,X,A,F,f\}$, which includes set of agent N, state space X, action set $A_i(x)$, agent cost function $F_i(x,a)$ and state transfer function $f(x,a)$.

1) State Space: For game agent $i\in N$, states $x_i=(v_i,s_i)$ are used to represent the current state of individual decision, where $v_i\in\mathbb{R}$ represents the current decision variables of agent i, $s_i\in\mathbb{R}$ represents the estimates of agent i for all global constraints. By constructing these states, agents can obtain virtual global information and continuously update their decisions and estimated information according to the state transfer function.

Each single-stage pumping station is considered an independent agent for the optimization scheduling of cascade pumping stations. The state of agent i is defined as: $x_i=\left(\{H_i^k,t_{i,m}^k\}_{k\in T,m\in M},\{e_i^k,\mu_i^k\}_{k\in T},\{c_{i,m}^k,\rho_{i,m}^k\}_{k\in T,m\in M}\right)$. $e_i^k\sim\sum_{i\in N}H_i^k-(Z_{out}^k-Z_{in}^k)-\Delta$h, where e_i^k denotes the estimate of agent i for the coupled constraint in the k^{th} period, μ_i^k is the Lagrange multiplier of agent i in the k^{th} period.

Each period is treated as an independent agent to handle the coupled constraint (2). Specifically, $c_{i,m}^k\sim\sum_{k\in T}\sum_{m\in M}\left(Q\cdot t_{i,m}^k\cdot 3600\right)-Q_{\mathrm{d}}$, where $c_{i,m}^k$ denotes the estimate of m^{th} water pump within agent i for the coupled constraint in the k^{th} period, $\rho_{i,m}^k$ is the corresponding Lagrange multiplier.

2) Actions: For agent $i\in N$, its action is defined as $a_i\in A_i(x_i)$, which can not only alter the true value of the decision from the previous game stage but also influence the estimated values of other agents. Specifically, for action $a_i=(\hat{v}_i,\hat{s}_i)$, where $\hat{v}_i\in\mathbb{R}$ is the change in the decision variable. $\hat{s}_i\in\mathbb{R}$ is the change in the estimates for constraints. For cascade pumping stations, actions are designed as follows:

$$a_i=\left(\{\hat{H}_i^k,\hat{t}_{i,m}^k\},\{\hat{e}_{i\to j}^k,\hat{\mu}_i^k\},\{\hat{c}_{i,m}^k,\hat{\rho}_{i,m}^k\}\right) \tag{8}$$

where $\hat{t}_{i,m}^k$ is the change in the running time for the m^{th} water pump of agent i in the k^{th} period, $\hat{H}_i^k$ is the change in head of agent i in the k^{th} period, $\hat{e}_{i\to j}^k$ represent the change in estimates that agent i passes to the j regarding the coupled constraint (3). similarly, $\hat{\mu}_i^k,\hat{c}_{i,m}^k,\hat{\rho}_{i,m}^k$ is the change in the corresponding states.

Additionally, $A_i(x_i)$ is the action set for agent i.

$$A_i(x)=\left\{\begin{array}{l}\{\hat{H}_i^k,\hat{t}_{i,m}^k\}_{k\in T,m\in M},\\ \{\hat{e}_{i\to j}^k,\hat{\mu}_i^k\}_{k\in T,j\in N_i},\\ \{\hat{c}_{i,m}^k,\hat{\rho}_{i,m}^k\}_{k\in T,m\in M},\\ H_{\min}^k\le H_i^k+\hat{H}_i^k\le H_{\max}^k,\\ t_{\min}^k\le t_{i,m}^k+\hat{t}_{i,m}^k\le t_{\max}^k\end{array}\right\} \tag{9}$$

where N_i represents the neighbor set of agent i.

3) State Transfer Function: The design of the state transfer function involves the influence of information interaction between agents on the dynamics of the game model. The new state value is updated according to the information from the previous stage. Interference effects on the state update process are ignored. The number of iterations is set as r.

(1)states of agents

$$H_i^k\left(r+1\right)=H_i^k\left(r\right)+\hat{H}_i^k(r) \tag{10}$$
$$t_{i,m}^k\left(r+1\right)=t_{i,m}^k\left(r\right)+\hat{t}_{i,m}^k(r) \tag{11}$$

(2)estimates of the coupled constraints

Each single-stage pumping station is mapped as an agent, information interaction between agents via $\hat{e}_{i\to j}^k$, and the state transfer function for constraint (3) is designed as follows:

$$e_i^k(r+1)=e_i^k(r)+\hat{H}_i^k(r)+\sum_{j\in N_i}\hat{e}_{j\to i}^k(r)-\sum_{j\in N_i}\hat{e}_{i\to j}^k(r) \quad (12)$$

$$\mu_i^k(r+1)=\mu_i^k(r)+\hat{\mu}_i^k(r) \quad (13)$$

Considering each period as an agent, we construct a fully connected, undirected communication topology graph between periods. The state transfer function for constraint (2) takes the following form.

$$c_{i,m}^k(r+1)=c_{i,m}^k(r)+Q\cdot\hat{t}_{i,m}^k(r)+T\cdot\hat{c}_{i,m}^k(r)-\sum_{\tau\in T}\hat{c}_{i,m}^\tau(r) \quad (14)$$

$$\rho_{i,m}^k(r+1)=\rho_{i,m}^k(r)+\hat{\rho}_{i,m}^k(r) \quad (15)$$

where τ represents the neighbor of the k^{th} period.

4) Invariance Property: The state transfer function ensures that coupled equality constraints are satisfied at any game stage. However, there are some limits for each agent. For the initial estimates of the state, The initial value of e must satisfy (16).

$$\sum_{i\in N}e_i^k(0)=\sum_{i\in N}H_i^k(0)-(Z_{out}^k-Z_{in}^k)-\Delta\text{h} \quad (16)$$

According to the action of the next stage, such as $a(1),a(2),...$, the state value is updated based on (17), and the subsequent state is satisfied $\sum_{i\in N}e_i^k(r) = \sum_{i\in N}H_i^k(r)-(Z_{out}^k-Z_{in}^k)-\Delta\text{h},\forall k\in T$.

$$x_i(r+1)=f(x_i(r),a(r)) \quad (17)$$

Therefore, the initial estimate of constraint (3) is given by $e_i^k(0)=H_i^k(0)-(Z_{i,out}^k-Z_{i,in}^k)-\Delta\text{h}/N$. Similarly, The initial value of c needs to be satisfied $\sum_{k\in T}\sum_{m\in M}c_{i,m}^k(0)=\sum_{k\in T}\sum_{m\in M}\left(Q\cdot t_{i,m}^k(0)\cdot 3600\right)-Q_\text{d},\forall i\in N$.

5) Agent Cost Function: The cost function refers to the payoff that each agent can obtain based on its chosen actions in the game. The information interaction between the agents influences the cost function of the game model. Specifically, for the optimization problem of cascade pumping stations, the agent cost function comprises three components. The first component is the agent's local objective function.

$$F_i^1(x,a)=\sum_{k\in T}\sum_{m\in M}\frac{\rho\cdot g\cdot Q\cdot H_i^k(r+1)\cdot t_{i,m}^k(r+1)}{1000\cdot\eta_{im}\cdot\eta_{mot}\cdot\eta_{dr}\cdot\eta_f}p^k \quad (18)$$

The second component addresses the effect of coupled constraints of the head.

$$\begin{aligned}F_i^2(x,a)=\sum_{k\in T}\Bigg\{&\left(e_i^k(r+1)\cdot\mu_i^k(r)-e_i^k(r)\cdot\mu_i^k(r+1)\right)\\&+\frac{1}{2}\left(e_i^k(r+1)^2\right)\\&+\frac{1}{2}\sum_{j\in N_i}\left(\mu_i^k(r+1)-\mu_j^k(r+1)\right)^2\Bigg\}\end{aligned} \quad (19)$$

The third component deals with the effect of coupled constraints on water diversion volume between each period.

$$\begin{aligned}F_i^3(x,a)=\sum_{m\in M}\Bigg\{&\sum_{k\in T}\left(c_{i,m}^k(r+1)\cdot\rho_{i,m}^k(r)-c_{i,m}^k(r)\cdot\rho_{i,m}^k(r+1)\right)\\&+\frac{1}{2}\sum_{k\in T}\left(c_{i,m}^k(r+1)^2\right)\\&+\frac{1}{2}\sum_{k\in T}\sum_{\tau\in T}\left(\rho_{i,m}^k(r+1)-\rho_{i,m}^\tau(r+1)\right)^2\Bigg\}\end{aligned} \quad (20)$$

The changes in $F_i^2(x,a)$ and $F_i^3(x,a)$ illustrate the convergence process of additional costs incurred by coupled equality constraints. The form of agent cost function for each single-stage pumping station is as follows.

$$F_i(x,a)=F_i^1(x,a)+F_i^2(x,a)+F_i^3(x,a) \quad (21)$$

B. Potential Function

In a state-potential game, both the agent cost function and the potential function have the same trend. The agent cost function characterizes the distributed decision mechanism in the optimization problem for each agent. The potential function shows the overall result of the agents' strategic choices in a game. The potential function guarantees that the global optimum of the optimization problem aligns with the stationary Nash equilibrium in the game model.

The potential function of the state-based potential game is the sum of three components: $J(x,a)=J^1(x,a)+J^2(x,a)+J^3(x,a)$. These components correspond to the respective parts of the agent cost function.

$$J^1(x,a)=\sum_{i\in N}\sum_{k\in T}\sum_{m\in M}\frac{\rho\cdot g\cdot Q\cdot H_i^k(r+1)\cdot t_{i,m}^k(r+1)}{1000\cdot\eta_{im}\cdot\eta_{mot}\cdot\eta_{dr}\cdot\eta_f}\cdot p^k \quad (22)$$

$$\begin{aligned}J^2(x,a)=\sum_{k\in T}\Bigg\{&\sum_{i\in N}\left(e_i^k(r+1)\cdot\mu_i^k(r)-e_i^k(r)\cdot\mu_i^k(r+1)\right)\\&+\frac{1}{2}\sum_{i\in N}\left(e_i^k(r+1)^2\right)\\&+\frac{1}{2}\sum_{i\in N}\sum_{j\in N_i}\left(\mu_i^k(r+1)-\mu_j^k(r+1)\right)^2\Bigg\}\end{aligned} \quad (23)$$

$$\begin{aligned}J^3(x,a)=\sum_{i\in N}\sum_{m\in M}\Bigg\{&\sum_{k\in T}\left(c_{i,m}^k(r+1)\cdot\rho_{i,m}^k(r)-c_{i,m}^k(r)\cdot\rho_{i,m}^k(r+1)\right)\\&+\frac{1}{2}\sum_{k\in T}\left(c_{i,m}^k(r+1)^2\right)\\&+\frac{1}{2}\sum_{k\in T}\sum_{\tau\in T}\left(\rho_{i,m}^k(r+1)-\rho_{i,m}^\tau(r+1)\right)^2\Bigg\}\end{aligned} \quad (24)$$

IV. DISTRIBUTED OPTIMIZATION ALGORITHM FOR GAME MODEL

A. Projected Gradient Algorithm

The game model provides an effective framework for distributed optimization problems. By designing reasonable algorithms, agents update their strategies according to their cost functions and available neighbor information. This paper chooses the projected gradient algorithm to calculate the change value of the action.

$$a_i(k)=\left[-\delta_i\cdot\frac{\partial F_i(x,a)}{\partial a_i}\bigg|_{a=0}\right]_{A_i(x)}^+ \quad (25)$$

where $[\centerdot]_{A_i(x)}^+$ is the projection of action values within the feasible range, and δ_i represents the step size of the agent i.

B. Stationary Nash Equilibrium

When the communication network topology is undirected and connected, each agent's estimates e and c satisfy the invariance property. If the constructed state-based potential game satisfies the following two conditions, the game model can converge to stationary Nash equilibrium [35].

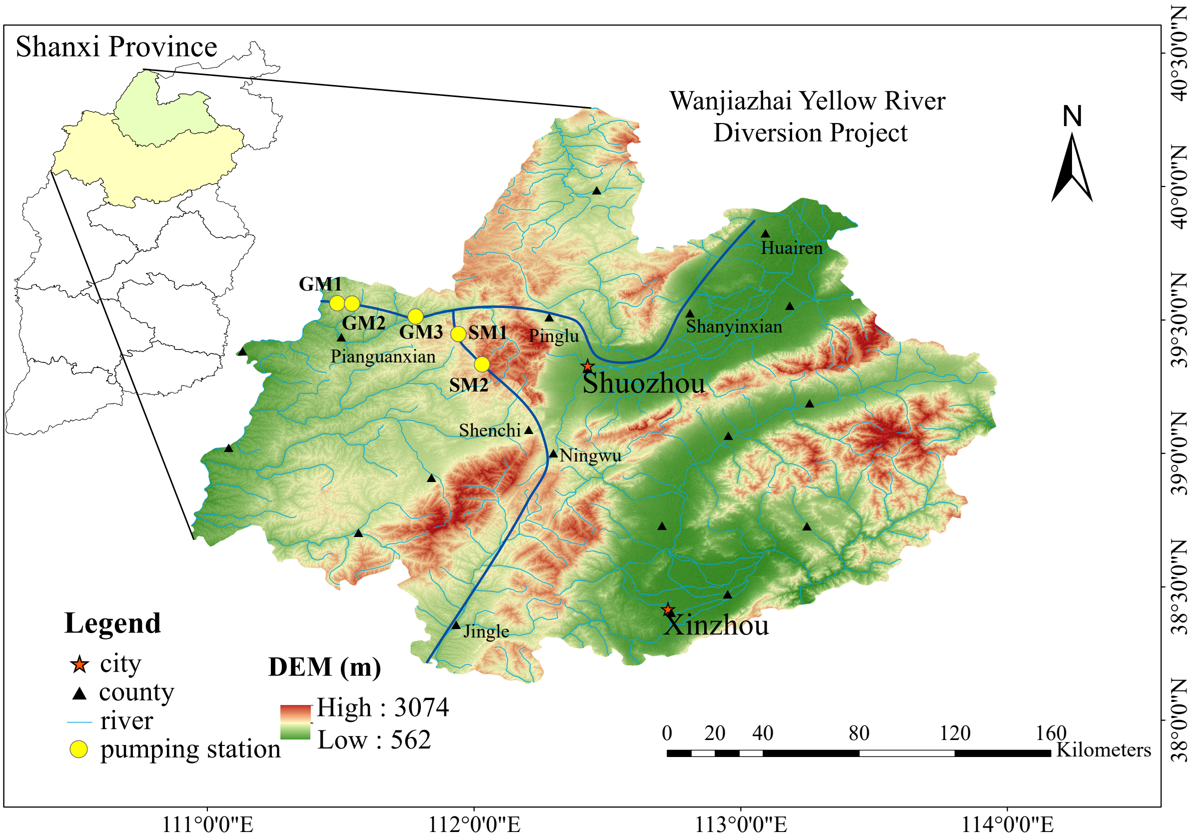

Fig. 2. The locations of Wanjiazhai Yellow River Diversion Project

TABLE I
OPERATIONAL PARAMETERS OF HEAD AND WATER LEVEL FOR CASCADE PUMPING STATIONS

Pumping station	Minimum head (m)	Design head (m)	Maximum head (m)	Forebay water level (m)	Afterbay water level (m)
GM1	106.53	140.00	163.00	971.05	1108.13
GM2	117.50	140.00	162.00	1106.89	1244.25
GM3	70.89	76.00	78.55	1218.69	1292.86

(1) For any agent $i \in N$, $a_i^* \in \arg\min_{a_{i \in A_i}} F_i(x^*, a_i, a_{-i}^*)$
(2) For both the previous state x^* and the next state $x = f(x^*, a^*)$, $x^* = x$ is satisfied.

Furthermore, the estimated values of information transmission satisfy $\hat{e}_{j\to i}^k(r) - \hat{e}_{i\to j}^k(r) = 0$ for all agents $i \in N$ and constraints $k \in T$. These changes demonstrate the equivalence between the stationary Nash equilibrium of the designed game and solutions to the coupled constrained optimization problem.

V. CASE STUDY

This section assesses the optimization performance of the proposed game model and algorithm in real-world cascade pumping stations. We conduct simulations on the MATLAB platform.

A. Parameter Settings

The Wanjiazhai Yellow River Diversion Project operates for 10 months each year, with an annual water diversion volume of 1.2×10$^9 m^3$. The project features three single-stage pumping stations along its main trunk lines: GM1, GM2, and GM3 [6]. Each single-stage pumping station contains 10 water pumps. Fig. 2 illustrates the specific water diversion route. For simplicity, we focus on fixed-speed water pumps with a flow rate of 6.45m^3/s to ensure efficient operation. The cascade pumping stations' specific water level and head parameters are detailed in Table I.

TABLE II
TIME-OF-USE ELECTRICITY PRICE IN SHANXI PROVINCE

Period Type	Period (h)	Time-of-use electricity price (Yuan/kWh)
Peak period	08:00-11:00 17:00-23:00	1.0499
Flat period	07:00-08:00 13:00-17:00 23:00-24:00	0.6963
Valley period	00:00-07:00 11:00-13:00	0.3722

According to the TOU electricity pricing in Shanxi Province, a day from 00:00 to 24:00 is divided into seven distinct peak and valley periods. The TOU electricity price is listed in Table II.

B. Optimization Results Analysis of Game Model

Firstly, we set the water delivery task of cascade pumping stations to 4×10$^6 m^3$. Based on the state-based potential game, the fully distributed optimization of the cascade pumping stations is designed. We chose network topology GM1 ↔ GM2 ↔ GM3. Each single-stage pumping station acts as an independent game agent to optimize its strategies when solving the coupled constraint problem locally.

1) Convergence of Agent Cost Function: The agent cost function $F_i^1(x,a)$ can clearly show the daily electricity operating cost changes for three single-stage pumping stations. The trend of $F_i^1(x,a)$ reveals how each single-stage pumping station adjusts its strategies to reduce its operating costs. The simulation results of agent cost function $F_i^1(x,a)$ for three single-stage pumping stations GM1, GM2, and GM3 are illustrated in Fig. 3.

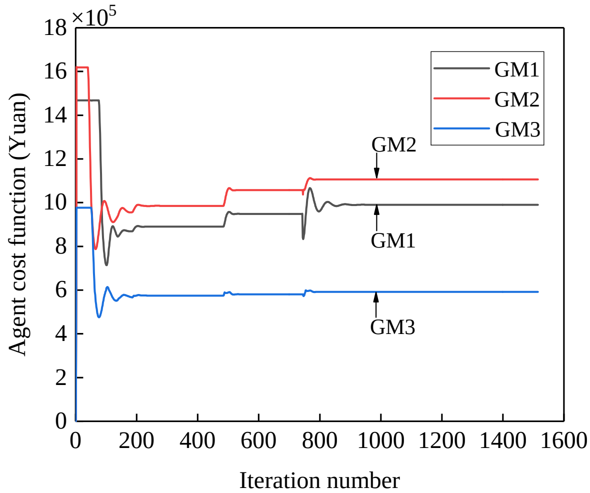

Fig. 3. Simulation results of agent cost function

The projected gradient algorithm converges to the optimal solution after 1516 iterations. The daily electricity operating costs for three single-stage pumping stations are: GM1 at 990,258.21 yuan, GM2 at 1,106,297.09 yuan, and GM3 at 592,015.04 yuan. There is a fluctuating increase in $F_i^1(x,a)$ around the 500th and 800th iterations. These changes indicate that the single-stage pumping stations are making a trade-off between minimizing the daily electricity operating costs and satisfying the coupled equality constraints.

To address the coupled equality constraints, we refer to the Lagrangian approach. We add additional cost functions $F_i^2(x,a)$and $F_i^3(x,a)$ to the objective function. These functions measure the extra costs of meeting the coupled equality constraints and show how the whole system converges to the global optimum.

When the game ends, $F_i^2(x,a)$and $F_i^3(x,a)$ gradually converge to zero. The system eventually complies with the coupled constraints. The different convergence values of the three agents are because of changes in head and running time. These results show that the proposed state-based potential game method can effectively optimize the operation of cascade pumping stations and reach the stationary Nash equilibrium.

2) Potential Function Analysis: The potential function helps us measure the change in costs of all agents with a unified criterion. By analyzing changes in the potential function, we can indirectly evaluate the system's overall performance and simplify complex system analysis.

As shown in Fig. 4, the potential function of cascade pumping stations gradually converges to 2,688,570.33 yuan. The $J^2(x,a)$ and $J^3(x,a)$ converge to zero. It indicates that global and local constraints are satisfied without additional virtual cost calculations. These results indicate that using state-based potential games to solve multi-agent system problems in cascade pumping stations is feasible and effective. Also, the convergence of the potential function proves that the stationary Nash equilibrium of the designed game is equivalent to the solutions of the optimization problem.

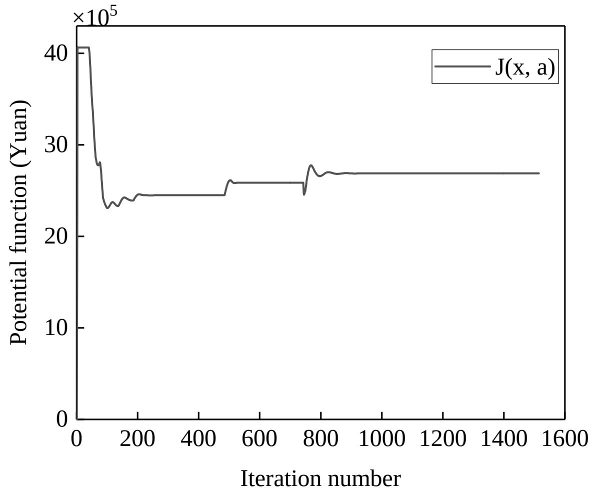

Fig. 4. Simulation results of potential function

3) Estimats of Coupled Equality Constraints: In the state-based potential game framework, the estimates of the Lagrange multipliers and constraints are exchanged between neighboring nodes. The estimated value can intuitively reflect whether the agents in the state-based potential game satisfy the coupled equality constraints through information exchange. Take the optimization process of the first pump in GM1 pumping station as an example. The simulation results of coupled equality constraints constraints (2) and (3) are shown in Fig. 5.

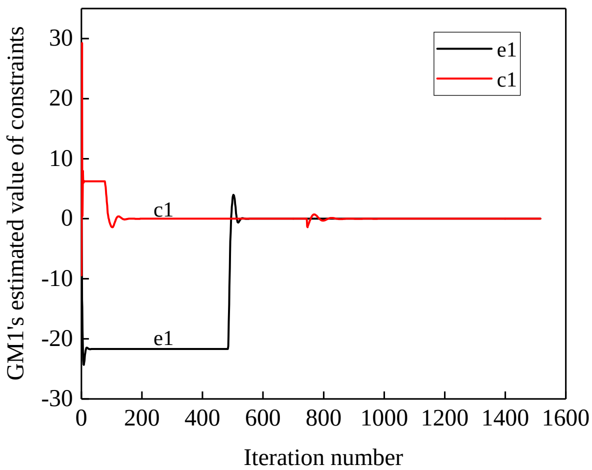

Fig. 5. Estimates of coupled equality constraints for game model

Specifically, e1 and c1 respectively represent the estimates of coupled equality constraint (3) and (2) of the first water pump in GM1 during the first period. The estimated value of e1 gradually converges to zero through information interaction via the communication topology between single-stage pumping stations. Considering each period in the single-stage pumping station as a virtual agent, the estimated value of c1 also converges to zero. These results show that the optimization strategies of cascade pumping stations meet the coupled equality constraints (2) and (3).

C. Evaluation of Pumping Station Operation Strategies

For three single-stage pumping stations, the head of each single-stage pumping station must satisfy the constraints (3), (5), (6) and (7). The simulation results for heads of GM1, GM2, and GM3 are shown in Table III. The head of the cascade pump station follows the upper and lower limit constraints. These scheduling strategies guarantee the normal and stable operation of cascade pumping stations.

TABLE III
THE HEAD OF PUMPING STATIONS AT DIFFERENT PERIODS

Period (h)	Period Type	GM1 head(m)	GM2 head(m)	GM3 head(m)
00:00-07:00	Valley	134.44	151.62	73.95
07:00-08:00	Flat	124.93	139.27	75.80
08:00-11:00	Peak	106.53	139.90	73.57
11:00-13:00	Valley	134.44	151.62	73.94
13:00-17:00	Flat	124.92	139.26	75.81
17:00-23:00	Peak	131.73	117.50	70.89
23:00-24:00	Flat	124.93	139.27	75.80

To minimize daily electricity operating costs, we must adjust the operation strategies of each single-stage pumping station dynamically according to TOU electricity pricing. When the electricity price is low, we can raise the single-stage pumping station's head and operating power to meet potential peak water demand. When the price is high, we should lower the single-stage pumping station's head and operating power to reduce unnecessary energy consumption.

To complete the daily water diversion task of 4×10$^6m^3$, GM1 adjusts the running time of 10 fixed-speed water pumps according to TOU electricity pricing.

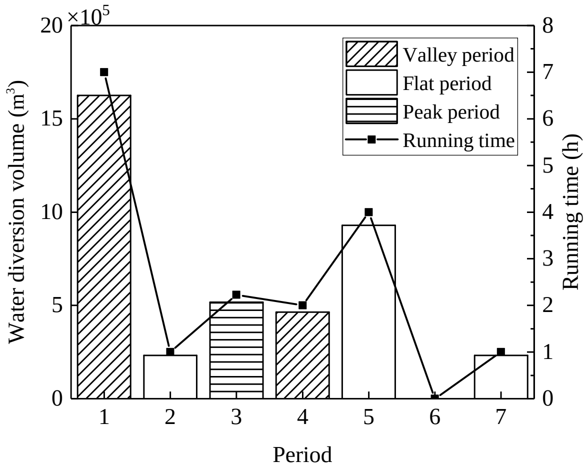

Fig. 6. The water delivery volume and running time at different periods

Fig. 6 shows the water diversion volume of the 7 periods and the running time of water pumps. The interval 0-1 represents the first period 00:00-07:00, and the rest follow the same rule. For the flat period and valley period, all 10 water pumps run throughout the period. During the peak period from 08:00-11:00, 10 water pumps in GM1 only run for 2.23 hours. And the single-stage pumping station stops running completely from 17:00 to 23:00.

The specific volume of water delivered in each period is as follows: 1,625,400 m^3, 232,200 m^3, 516,999.96 m^3, 464,400 m^3, 928,800 m^3, 0 m^3, and 232,200 m^3. The results show that cascade pumping stations can effectively adjust water pump strategies according to TOU electricity pricing. TOU electricity pricing encourages efficient energy consumption patterns, alleviates peak load pressures, and ultimately leads to cost savings by shifting usage to off-peak periods. This approach maximizes the economic benefits of electricity pricing mechanisms. It helps us better optimize resource allocation and enhance the overall system efficiency of cascade pumping stations.

D. Adaptation to Variations in Water Diversion Volume

Because the daily water diversion volume of cascade pump stations is substantial, there are different water diversion tasks on other days. The operation plan requires completing a daily water delivery task of 3.8×10$^6m^3$, and the equality constraint of the head remains unchanged. This experiment evaluates how well the algorithm performs under different conditions to show its feasibility and rationality.

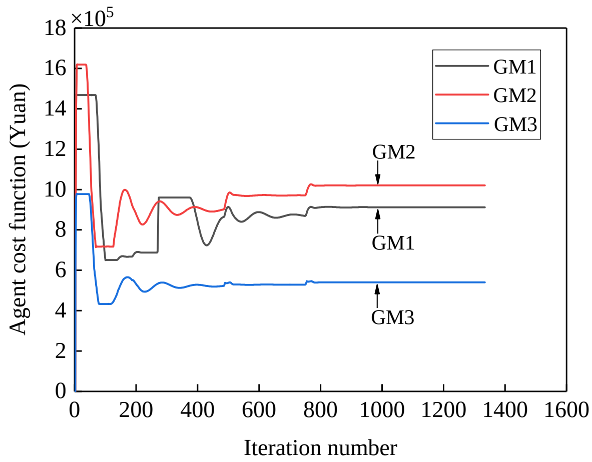

Fig. 7. Agent cost function under different water delivery task

The daily electricity operating costs for three single-stage pumping stations are as follows: GM1 at 912,134.85 yuan, GM2 at 1,020,146.44 yuan, and GM3 at 540,039.18 yuan. The potential function of the cascade pumping stations gradually converges to 2,472,320.48 yuan. At the same time, the additional cost functions $F_i^2(x,a)$and $F_i^3(x,a)$ also converge to 0. The water volume affects the daily electricity operating costs of the cascade pumping station. When the water delivery task decreases by 2×10$^5m^3$, the daily electricity operating costs decrease by 216,249.85 yuan. The results show that the distributed projected gradient method based on the state-based potential game can solve various water diversion volume optimization problems.

Due to the water diversion task change, the running time of single-stage pump stations is reduced from 2.23 hours to 1.73 hours during the 08:00-11:00 peak period. Shortening the running time can greatly decrease the daily electricity operating costs for single-stage pumping stations.

The running time of the water pump remains unchanged during other periods. For head constraints, as only the water volume constraint changes, the heads for GM1, GM2, and GM3 remain unchanged.

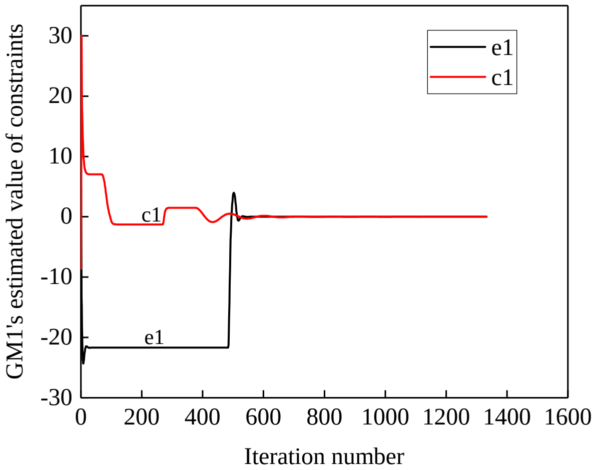

Fig. 8. Estimates of coupled equality constraints under different water delivery task

Similarly, e1 and c1 represent estimates of couled equality constraint (3) and (2) for the first water pump in GM1 during the first period, respectively. Both the estimated values of e1 and c1 converge to zero. This shows that the optimization strategies employed by the cascade pumping stations satisfy the equality constraints (3) and (2).

E. Assessment of Variations in Network Topology

In the above experiments, the network topology used for distributed optimization is GM1 $\leftrightarrow$ GM2 $\leftrightarrow$ GM3. The information needed to be shared among the neighboring agents is the estimates of the coupled equality constraints $\{e_i^k, \mu_i^k\}$ and their Lagrange multipliers $\{c_{i,m}^k, \rho_{i,m}^k\}$ as well as the corresponding actions. Local communication between agents only transmits the estimated information of the neighbors. None of them reveals detailed information on the preferences, economic factors, or loads of the users. Thus, the privacy of each user is well preserved.

In order to analyze convergence under different network topologies, the network topology changes from linear to ring topology. The specific change is shown in Fig. 9.

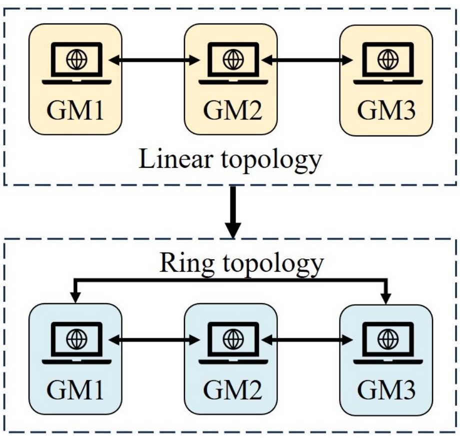

Fig. 9. Changes in network topology

The water delivery task remains at $4\times10^6 m^3$. Fig. 10 shows the simulation results of the agent cost function under ring topology.

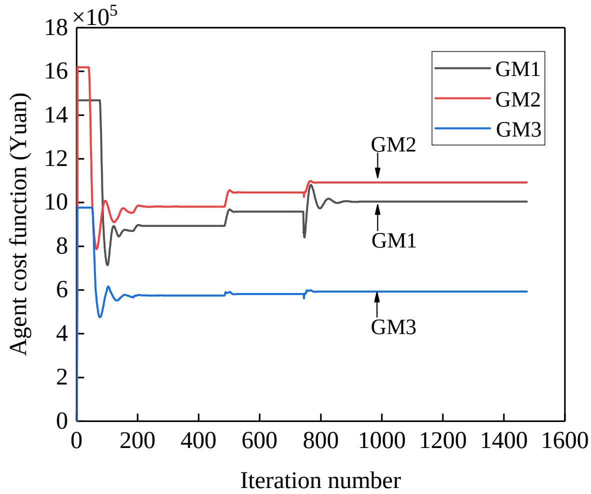

Fig. 10. Agent cost function under ring topology

The agent cost function $F_i^1(x, a)$ under ring topology converges to the optimal solution after 1476 iterations. Because the water delivery task remains unchanged, the daily electricity operating cost of the cascade pumping stations, i.e. the potential function, is 2,688,570.33 yuan. The daily electricity operating costs for the three single-stage pumping stations have changed: GM1 at 1,004,099.88 yuan, GM2 at 1,091,849.68 yuan, and GM3 at 592,620.77 yuan.

In distributed optimization algorithms, the convergence speed increases as individual agents receive more information. However, the closed loop by agents GM1, GM2, and GM3 introduces additional communication paths and increases communication volume. This can result in information overload and redundant calculations, which partly offset the acceleration effect. Consequently, the added communication paths do not significantly improve overall algorithm performance.

By comparing Table III and Table IV, the head of GM1 and GM3 increases, while the head of GM2 decreases. Because the communication topology has changed, GM1 and GM3 can exchange information. GM2, as an intermediate node, reduces the communication burden and balances the head of cascade pumping stations better. And the change in head is also adjusted according to TOU electricity pricing.

TABLE IV
HEAD OF PUMPING STATIONS UNDER RING TOPOLOGY

Period (h)	Period type	GM1 head(m)	GM2 head(m)	GM3 head(m)
00:00-07:00	Valley	137.45	148.42	74.12
07:00-08:00	Flat	126.59	137.56	75.84
08:00-11:00	Peak	106.53	139.62	73.85
11:00-13:00	Valley	137.45	148.42	74.12
13:00-17:00	Flat	126.59	137.56	75.85
17:00-23:00	Peak	131.67	117.50	70.89
23:00-24:00	Flat	126.59	137.56	75.85

In conclusion, the original linear topology, where agents GM1, GM2, and GM3 are connected sequentially, allows information to pass only between adjacent nodes. This setup aligns with the real-world geographical layout of long-distance water diversion projects. For the optimization scheduling problem of cascade pumping stations, this approach reflects realistic spatial limits and ensures optimal strategies can be found with limited information exchange.

VI. CONCLUSION

The proposed state-based potential game model for cascade pumping stations provides an effective solution for the distributed optimization scheduling of water pumps. By considering the interactions between single-stage pumping stations, we designed corresponding states and estimates to handle coupled equality constraints. Subsequently, we introduce the potential function with good convergence properties to unify the cost function of multiple agents into a global function. It simplifies the analysis process of the game model. Finally, the projected gradient algorithm is used to achieve stationary Nash equilibrium. Based on TOU electricity pricing, the results demonstrate that cascade pumping stations can efficiently complete water diversion tasks while minimizing daily electricity operating costs. The proposed algorithm shows excellent convergence performance. In addition, we also test the game model under different water diversion tasks and network topologies. The results confirm the feasibility and rationality of the state-based potential game framework for the distributed optimization of cascade pumping stations.

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