Black-Litterman Portfolio Optimization Using Gaussian Mixture Model

Liangyu Min*, *Member*, *IAENG*, Dijia Lin, Jianan Wang, Bolin Wang

Abstract—**The Black-Litterman portfolio model based on the Gaussian mixture model is proposed in this study. Different from current popular research patterns in Black-Litterman portfolio, we do not employ some forecasting algorithms to build investment views, but use the clustering method to dig the historical samples and find the representative centroids as the investment experts. Due to GMM believes that the data follows a mixed normal distribution, with each component is assigned a probability value. We build three types of Black-Litterman portfolio based on the designed selection algorithm, the optimistic-style GMM-BL, the pessimistic-style GMM-BL, and the maximum probability GMM-BL. The corresponding numerical experiments focus on testing the out-of-sample performance of the proposed portfolio models and the baseline strategy, where the maximum probability GMM-BL and the pessimistic-style GMM-BL achieve the highest Sharpe ratio and Calmar ratio, and the optimistic-style GMM-BL shows similar performance with the $1/N$ model. Although the proposed framework tends to be conservative, the out-of-sample effectiveness still demonstrates its practical value.**

Index Terms—**Portfolio selection, Gaussian Mixture Model, Black-Litterman, Clustering method**

I. Introduction

THE Black-Litterman (BL) portfolio model[1], [2], [3] has been recognized as one of the practical investment strategies by academia and industry[4], [5], [6], [7], [8], [9], [10], [11], because of its outstanding ability to overcome some notorious drawbacks of the classical mean-variance (MV) portfolio model[12], [13]. For example, MV only uses historical samples to generate model parameters, which resulting severe parameter-dependency and high parameter-sensitivity. Thus, it is difficult for MV to achieve impressing performance in the out-of-sample numerical experiments. Black-Litterman designs a ingenious way to reduce the parameter sensitivity to some extent, where the investor views can be used to fix the estimated parameters from the historical samples. Therefore, Black-Litterman portfolio model generally shows better out-of-sample performance than the classical MV model, as long as the inputted investor views are reliable.

Manuscript received Jul 17, 2024; revised Feb 13, 2025. This work is supported by the College Young Teacher Training Subsidy Plan of Shanghai (No. AG24-33811-3301).

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The key insight of the Black-Litterman method is to blend the prior market portfolio with the subjective investor views via a Bayesian approach, where the Gaussian assumption is hold for model derivation process. Some scholars also develop non-Gaussian Black-Litterman portfolio model. Meucci[3] considers non-normal investor views, and combines arbitrarily distributed market prior with these views in a manner of copula method. Sahamkhadam et, al.[8] extends the Black-Litterman framework to incorporate tail dependency, and the joint posterior returns distribution is estimated by vine copulas. The associated empirical analysis demonstrates the robustness of the copula-based BL portfolio framework, especially in terms of lower tail risk. Recently, artificial intelligence (AI) has showcased enormous potential in financial modeling, some machine learning and deep learning algorithms proven to be appropriate for giving provident predictions, which are beneficial in building AI-based BL portfolio model[14], [15], [16], [17], [18].

Another strand of research route is to combine some econometric methods, statistical approaches, and time-series analysis with the traditional BL framework to improve portfolio performance[19], [5], [9], [11], [10]. The core idea of these variants is to empower the forecasting ability for the constructed strategies, which is equal to achieve high-quality expert investment views in the classical BL method. Nonetheless, how to specify the uncertainty level of the simulated investor views is still one of the most complicated issues discussed in academic communities. He & Litterman (2002) [20] directly set the uncertainty level (quantified by variance) of investor views to be proportional to the uncertainty level of the prior market equilibrium portfolio. Idzorek (2007) [21] quantifies the uncertainty level of the investor view using a percentage between 0% and 100%, where 0% means the investor view can not be believed while 100% indicates the given investor view is highly reliable. Although Idzorek provides a feasible solution to adjust the BL posterior returns distribution according to the given confidence level, but the specific method to set the confidence level for each investor view is not mentioned. Li et, al. (2023) [22] quantify the investment view uncertainty based on theory of Gaussian process regression, which is consistent with the Black-Litterman framework in spirit of basic assumption. To eradicate the arbitrariness and subjectivity in viewpoint setting as much as possible, we develop the Gaussian Mixture Model based Black-Litterman (GMM-BL) portfolio model in this work, where the normality assumption is hold throughout the paper.

Combine optimization techniques and machine learning methods in portfolio formation gradually become the mainstream of financial modeling research community[23], [24], [25], [26], [10]. Existing numerical experiments[7] illustrate that direct forecasting asset return rates may not be very robust in the out-of-sample performance, in that the issue of over-fitting in lots of tree-based models and neural network-based systems. Scholars and practitioners have proposed some impressing strategies to cope with this problem. For example, one can call multiple algorithms to simulate different investment experts simultaneously, and design an ensemble mechanism to evaluate these investor views. In this paper, we do not consider many algorithms to give predictions, because AI-based methods of identical types tend to present similar results, whereas the complicated underlying structure of asset returns might not given enough attention.

In order to analyze financial data at a finer granularity and build Black-Litterman portfolio model, we take data as variables following a set of multivariate normal distributions, and the Gaussian Mixture Model is used to cluster the samples. By this way, both the impact of macroeconomics and the technical indicators can be taken into account for simulating investor opinions. Essentially, the proposed portfolio model combines BL framework with clustering method[27], [28], [25], contributing to provide a robust approach for investment views generation, with quantifiable uncertainty. In principle, Gaussian mixtures can approximate any continuous distribution[29], whose components can be interpreted as market regimes. Therefore, some non-normal attributes of financial returns such as skewness and kurtosis can be modeled by GMM, which also supports the rationality of the proposed GMM-BL portfolio model.

The rest of this study is structured as follows. Section II summaries the classical Black-Litterman portfolio framework, which is the basis for this research. Section III provides the derivation of the Gaussian Mixture Model, and the associated GMM-BL portfolio is constructed and presented in Section IV. In Section V, the proposed strategies and benchmarks are tested on the real-world datasets, and the corresponding analysis are given. Section VI draws a conclusion of the whole paper and discusses the possible limitations of this study at last.

Note that the lower case bold letters refer to vectors, for example, $\boldsymbol{a}$. The upper case bold letters denote matrices such as $\mathbf{A}$. Meanwhile, scalars such as A are indicated by the plain white letter.

II. Black-Litterman Framework

Analytically, define a universe of N risky assets over time period of length T, whose $N \times T$ return matrix $\mathbf{R} = (\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n)^T$ follows the multivariate Gaussian distribution, $\boldsymbol{r} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{r}] = (\mu_1, \mu_2, \ldots, \mu_n)$ represents the expected asset return, and $\boldsymbol{\Sigma}$ is the covariance matrix.

Under the assumption of Gaussian, the expected return $\boldsymbol{\mu}$ is deemed to be composed of market equilibrium return $\boldsymbol{\Pi}$ and Gaussian residual vector $\boldsymbol{\varepsilon}_\mu \sim \mathcal{N}(\mathbf{0}, \tau\boldsymbol{\Sigma})$. Therefore, we have $\boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma})$, which constitutes the prior distribution of the involved assets. The market equilibrium return $\boldsymbol{\Pi}$ can be achieved using reverse optimization, where each investor holds the same market portfolio with the utility objective of $\max \boldsymbol{x}^T\boldsymbol{\Pi} - \lambda \boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{x}$, where $\boldsymbol{x} \in \mathbb{R}^n$ is the portfolio weight vector, and $\lambda \in (0,1)$ is the risk-aversion coefficient. By solving the maximization problem, we have $\boldsymbol{\Pi} = \lambda\boldsymbol{\Sigma}\boldsymbol{x}, \lambda = \frac{r_m - r_f}{\sigma_m^2}$, where r_m is the market return rate, r_f is the risk-free rate, and σ_m^2 is the variance of market return rate. τ is a scalar representing the uncertainty of the prior distribution, which is proportional to the confidence level of the market equilibrium prior.

Black-Litterman framework takes the subjective investor views as the likelihood estimations for the underlying return distribution, and can fix the market prior via the Bayesian formula. Assume there are K investor views about $M(M \leq N)$ risky assets, the pick matrix $\mathbf{P} \in \mathbb{R}^{K \times M}$ and the investor views vector $\mathbf{Q} \in \mathbb{R}^K$ can be used to describe the subjective views as follows:

$$\mathbf{P}\boldsymbol{\mu} = \mathbf{Q} + \boldsymbol{\varepsilon}_v, \boldsymbol{\varepsilon}_v \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Omega}), \mathrm{Cov}(\boldsymbol{\mu}, \boldsymbol{\varepsilon}_v) = 0$$

Thus, we have $\mathbf{P}\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{Q}, \boldsymbol{\Omega})$, and the posterior distribution of $\boldsymbol{\mu}$ can be achieved via Bayesian formula as follows:

$$\mathcal{N}([(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}^T\boldsymbol{\Omega}\mathbf{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \mathbf{P}^{-1}\boldsymbol{\Omega}^{-1}\mathbf{Q}], [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}^T\boldsymbol{\Omega}\mathbf{P}]^{-1})$$

where

$$\hat{\boldsymbol{\mu}} = (\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}^T\boldsymbol{\Omega}\mathbf{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \mathbf{P}^{-1}\boldsymbol{\Omega}^{-1}\mathbf{Q}$$
$$\hat{\boldsymbol{\Sigma}} = (\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}^T\boldsymbol{\Omega}\mathbf{P}]^{-1}$$

Based on the posterior distribution, the Black-Litterman portfolio can be expressed as follows:

$$\max_{\boldsymbol{x}} \boldsymbol{x}^T\hat{\boldsymbol{\mu}} - \frac{\lambda}{2}\boldsymbol{x}^T\hat{\boldsymbol{\Sigma}}\boldsymbol{x} \tag{1}$$

An illustrative example is given as follows to present the basic logic of the investor views in Black-Litterman framework. Assume there are four available risky assets named A, B, C, D on the market. An investment expert has the following three opinions about the involved assets:

- The yield of asset A is 20%.
- The yield of asset B is 10% higher than that of asset C.
- The yields of assets A and C are 5% higher than that of asset D.

Therefore, the pick matrix $\mathbf{P}$ is as follows:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1/2 & 0 & 1/2 & -1 \end{bmatrix}$$

and the associated views vector $\mathbf{Q} = [20\%, 10\%, 5\%]^T$ with the $K \times K$ covariance matrix $\boldsymbol{\Omega} = \mathrm{diag}(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}^T)$.

III. Gaussian Mixture Model

Gaussian Mixture Model (GMM) belongs to the probabilistic model used for representing the presence of subpopulations within an overall population. The fundamental assumption of GMM is that the data is generated from a mixture of several Gaussian distributions. Each Gaussian component indicates a cluster within the data. The overall distribution is a weighted sum of these Gaussian components. Assume that each cluster C_i can be represented by a multivariate Gaussian distribution as follows:

$$
\begin{aligned}
f_i(\boldsymbol{x}) &= f(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) \\
&= \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}_i|^{1/2}} \exp\left\{ -\frac{(\boldsymbol{x}-\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i)}{2} \right\}
\end{aligned}
$$

where $f_i(\boldsymbol{x})$ is the probability density of $\boldsymbol{x}$ belonging to cluster C_i. For the population $\mathbf{X}$, we assume its probability density function is a GMM of k clusters:

$$
f(\boldsymbol{x}) = \sum_{i=1}^{k} f_i(\boldsymbol{x})P(C_i) = \sum_{i=1}^{k} f_i(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)P(C_i)
$$

where $P(C_i)$ is the mixture parameter, satisfying the constraint $\sum_{i=1}^{k} P(C_i) = 1$ for total k clusters. Hence, the parameters of GMM is defined as follows:

$$
\boldsymbol{\theta} := \{\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, P(C_1), \ldots, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k, P(C_k)\}
$$

A. Maximization Likelihood Estimation

Generally, given data set $\mathbf{D}$, the likelihood function for $P(\mathbf{D}|\boldsymbol{\theta})$ is as follows:

$$
P(\mathbf{D}|\boldsymbol{\theta}) = \prod_{j=1}^{n} f(\boldsymbol{x}_j)
$$

where the assumption of independent and identically distributed (i.i.d) samples holds. And, the corresponding log-likelihood function is as follows:

$$
\begin{aligned}
\ln P(\mathbf{D}|\boldsymbol{\theta}) &= \sum_{j=1}^{n} \ln f(\boldsymbol{x}_j) \\
&= \sum_{j=1}^{n} \ln\left(\sum_{j=1}^{n} f(\boldsymbol{x}_j|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)P(C_i) \right)
\end{aligned}
$$

Note that it is hard to solve $\boldsymbol{\theta}$ analytically, but it can be approximately estimated by the Expectation-Maximization (EM) algorithm.

B. Expectation-Maximization Algorithm

EM algorithm iteratively updates $\boldsymbol{\theta}$ to maximize the likelihood of the observed samples. Initially, for each cluster C_i, $\boldsymbol{\mu}_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{id})^T$ can be generated by uniformly sampling from every dimension, with $\boldsymbol{\Sigma}_i = \mathbf{I}$. The prior probability for each cluster is $P(C_i) = \frac{1}{k}$.

The E-step aims to evaluate the weight w_{ij} (contribution) of data point $\boldsymbol{x}_i$ to cluster C_j, that is, $\mathbb{E}[c_{ji}] = P(C_i|\boldsymbol{x}_j) = w_{ij}$, which can be calculated as follows:

$$
w_{ij} = P(C_i|\boldsymbol{x}_j) = \frac{P(\boldsymbol{x}_j|C_i)P(C_i)}{P(\boldsymbol{x}_j)} = \frac{f(\boldsymbol{x}_j|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)P(C_i)}{f(\boldsymbol{x}_j)}
$$

The M-step gives the likelihood estimation of $\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i, P(C_i)$ using the obtained w_{ij} in the E-step:

$$
\begin{cases}
\boldsymbol{\mu}_i = \frac{\sum_{j=1}^{n} w_{ij}\boldsymbol{x}_j}{\sum_{j=1}^{n} w_{ij}} \\
\boldsymbol{\Sigma}_i = \frac{\sum_{j=1}^{n} w_{ij}(\boldsymbol{x}_j-\boldsymbol{\mu}_i)(\boldsymbol{x}_j-\boldsymbol{\mu}_i)^T}{\sum_{j=1}^{n} w_{ij}} \\
P(C_i) = \frac{\sum_{j=1}^{n} w_{ij}}{n}
\end{cases}
$$

IV. GMM-BL Portfolio Model

The theoretical basis for our proposed GMM-BL model is that history would repeat itself, especially on the scope of finance and economics. According to this hypothesis, we can firstly extract the local trends from market data points as the k clusters, representing the multiple investors opinions at different periods. Then, we use the obtained k components to evaluate each stock, and get the forecasting return rates. Accordingly, the confidence level can be quantified by the mixture parameter $P(C_i)$. Algorithm 1 presents the basic logic of GMM-BL.

Algorithm 1 Algorithm of GMM-BL portfolio model.

Input: Market data $\mathbb{D}_m$; Asset data $\mathbb{D}_a$.

Output: GMM-BL portfolio; Pick matrix $\mathbf{P}$; Investor views $\mathbf{Q}$; Confidence matrix $\boldsymbol{\Omega}$.

1: Separate training samples from historical datasets, i.e., $\mathbb{T}_m$ for market data and $\mathbb{T}_a$ for asset universe.
2: Define the number of components k for the GMM according to the overall trend of $\mathbb{T}_m$.
3: For each stock in $\mathbb{T}_a$, get k investor views as well as $P(C_i)$ using GMM.
4: Construct the confidence matrix based on $P(C_i)$.
5: Construct the pick matrix $\mathbf{P}$.
6: Construct the views vector $\mathbf{Q}$ based on the GMM.
7: Obtain the posterior distribution and calculate the parameters $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$.
8: Build the GMM-BL portfolio model based on (1).
9: **return** GMM-BL.

Some practical constraints are also considered in the proposed GMM-BL model, the first one is non-shorting constraint, $\boldsymbol{x} \geq \mathbf{0}$, and the second one is budget constraint, $\mathbf{1}^T\boldsymbol{x} = 1$. The objective of the proposed GMM-BL portfolio is to maximize Sharpe ratio[30], where both the investment return and portfolio risk are taken into account:

$$
\begin{aligned}
\max_{\boldsymbol{x}} \quad & \frac{\boldsymbol{x}^T\hat{\boldsymbol{\mu}}}{\sqrt{\boldsymbol{x}^T\hat{\boldsymbol{\Sigma}}\boldsymbol{x}}} \\
s.t. \quad & \boldsymbol{x} \geq \mathbf{0}, \mathbf{1}^T\boldsymbol{x} = 1
\end{aligned} \tag{2}
$$

Note that the problem belongs to nonlinear programming, which can be solved by open-source modules such as SciPy[31], or commercial solvers such as Gurobi and CPLEX.

V. Numerical Experiments

The numerical experiments provide model performances on the realistic datasets for analyzing the efficient and effectiveness of the proposed GMM-BL portfolio. We use the equal-weighted portfolio as the benchmark, in that it has been demonstrated to show stable out-of-sample performance [32]. Also, we focus on evaluating portfolio out-of-sample performance, in which some risk-adjusted indicators such as Sharpe ratio and Calmar ratio are involved. Note that, the risk-free rate $r_f = 3\%$ per year throughout the numerical experiments.

TABLE I
FINANCIAL INFORMATION OF THE INVOLVED CORPORATIONS.

Ticker	Corporation	Market Capital	β(5 Year Monthly)	PE ratio (TTM)	EPS (TTM)
AAPL	Apple Inc.	3.001T	1.25	30.49	6.42
BA	The Boeing Company	115.609B	1.55	-	-3.53
C	Citigroup Inc.	117.886B	1.47	18.02	3.43
GE	General Electric Company	176.057B	1.24	42.33	3.80
GOOG	Alphabet Inc.	2.184T	1.01	27.25	6.52
INTC	Intel Corporation	129.579B	1.06	31.38	0.97
JPM	JPMorgan Chase & Co.	563.49B	1.11	11.84	16.57
MSFT	Microsoft Corporation	3.141T	0.89	36.65	11.53
NVDA	NVIDIA Corporation	2.953T	1.69	70.03	17.14
TSLA	Tesla Inc.	559.00B	2.32	44.94	3.90

Acronym: 3.001T indicates 3.001 trillion; 115.609B means 115.609 billion; TTM: Trailing Twelve Months.

TABLE II
DESCRIPTIVE STATISTICS OF THE INVOLVED ASSETS RETURN RATES.

Ticker	Mean	Stdev.	Jarque-Bera	p-value
AAPL	0.0016	0.0215	840.3486	0.00
BA	0.0001	0.0345	4817.7620	0.00
C	0.0002	0.0264	4614.3127	0.00
GE	0.0009	0.0264	4614.3127	0.00
GOOG	0.0009	0.0208	546.3112	0.00
INTC	0.0000	0.0248	4779.2096	0.00
JPM	0.0006	0.0219	5523.1220	0.00
MSFT	0.0012	0.0204	1886.8853	0.00
NVDA	0.0026	0.0333	239.2627	0.00
TSLA	0.0033	0.0431	395.9242	0.00
SPX	0.0005	0.0146	5670.8814	0.00

TABLE III
TECHNICAL INDICATORS.

Item	Detail	Type
ADX	Average Directional Movement Index	Momentum
ADOSC	Chaikin A/D Oscillator	Volume
NATR	Normalized Average True Range	Volatility
MIDPRICE	Mid Point Over period	Overlap

A. Data Set

We collect daily fundamental trading information of 10 giant corporations on the US equities market from the database of Finance Yahoo, including open price, high price, low price, close price, adjusted close price, and volume. Table I summarizes the fundamental information of the involved companies, where MSFT has the highest market capital of 3.141 trillion, TSLA shows the largest β of 2.32, indicating significant higher level of volatility than market. PE ratio suggests insight into how much investors are willing to pay for each dollar of earnings, that is, the market expectation for the company. Note that NVDA achieves the highest PE ratio of 70.03, whereas BA has the negative EPS of -3.53, hence its PE ratio is meaningless. Earning Per Share (EPS) is a financial metric illustrating the profitability of a company on a per-share basis. Investors can use this indicator to gauge a company's financial health and performance. NVDA also obtains the best EPS of 17.14, followed by JPM, of 16.57.

Table II gives the descriptive statistics of the involved corporations as well as the S&P 500 index (SPX), where the mean value, standard deviation are provided. We also present the results of Jarque-Bera statistical test for each asset, and no normality could be observed according to the associated p-values (the null hypothesis is rejected at the significance level of 0.1%). The SPX figure on the website of Yahoo Finance visualizes the market trend in a manner of candlestick chart, where the training period ranging from May, 2019 to May, 2023. Significantly, different market regimes can be observed, which provide practical basis for GMM modeling.

B. Technical Indicators

Besides the fundamental financial features, we also make some technical indicators for better capturing the characteristics of these assets. Table III illustrates the used technical indicators, where the short period is 5 days and the long period is 10 days in the associated computations.

C. Evaluation Measures

For the purpose of comparing different strategies comprehensively, we employ some performance measures, namely return on investment (ROI), annual percentage yield (APY), maximum drawdown (MDD), Sharpe ratio (SR), and Calmar ratio (CR). Define R_T as the final wealth obtained by a investment strategy, and R_0 as the initial wealth, ROI is given as follows:

$$\text{ROI} = \frac{R_T - R_0}{R_0} \times 100\%$$

Based on that, APY is calculated by the following formula:

$$\text{APY} = \sqrt[n]{1 + \text{ROI}} - 1$$

where n is the year of investment period, and we assume 252 trading days per year.

MDD can be used to measure the portfolio downside risk, which is define as the peak-to-trough decline of a investment over a specified time period, which can be expressed by $\text{MDD} = \max_{t\in[0,T]}\{\max_{i\in[0,t]} \text{ROI}_i - \text{ROI}_t\}$. Generally, MDD is quoted as a ratio of the peak value:

$$\text{MDD} = \max_{t\in[0,T]} \left\{ \frac{\max_{i\in[0,t]} \text{ROI}_i - \text{ROI}_t}{\max_{i\in[0,t]} \text{ROI}_i} \right\} \times 100\%$$

where $\max_{i\in[0,t]} \text{ROI}_i$ documents the highest peak from the initial point to the instant t.

SR is a widely used indicator in appraising investment strategy, measuring the excess return obtained with the predefined portfolio risk.

$$\text{SR} = \frac{\text{APY} - R_f}{\sigma_p}$$

where R_f is the risk-free rate, σ_p is the annual standard deviation (VOL) of the investment return rates. Conventionally, scholars take the Treasury bill rate or long-term government bond yield as the risk-free rate in practice. In this study, we set $R_f = 3\%$ per year for simplicity.

CR is also a risk-adjusted financial measure advocated by hedge fund managers. Slightly different from SR, the denominator in CR is MDD instead of VOL, suggesting it focuses on the portfolio downside risk.

$$\text{CR} = \frac{\text{APY}}{\text{MDD}}$$

D. Investor views

According to the overall status shown in Yahoo Finance ranging from June 2019 to May 2023, three identifiable sub-trends can be found and simulated as investment experts. Table IV reports the results of executing algorithm 1 on the practical data set. BA seems not to be favored by these experts, due to its low EPS and unsatisfactory average return rate. Whilst NVDA and GE are ranked high positions by these experts, where all of forecasting returns are positive.

Based on that, we construct three types of GMM-BL portfolios: the first one is optimistic investment style, where the view with highest possible return is adopted for building subsequent portfolio; the second one is pessimistic investment style, where the view with lowest possible return is used for constructing Black-Litterman model; the last one portfolio takes the forecasting return rate with the most confidence level as the investor view to form investment strategy.

E. Portfolio Performance

The risk-aversion coefficient $\lambda = 1.9868$ can be calculated by the historical prices of SPX, and the correlation matrix can be achieved via the Ledoit-Wolf estimation method, which is a robust and efficient manner, particularly when the number of observations is not much larger than the number of assets. Essentially, this approach aims to improve the stability and accuracy of the covariance matrix by shrinking the sample covariance matrix towards a more structured estimator, often the identity matrix or a constant correlation matrix. In portfolio optimization, a more stable covariance matrix leads to better estimates of portfolio risk and more reliable results.

The correlation matrix suggests that JPM and Citi have similar stock trends, since the correlation coefficient between them is around 0.9, and both of them belong to financial industry. Also, the correlation coefficients between MSFT & AAPL, MSFT & GOOG are above 0.6. Whilst GE and TSLA show low correlation coefficient around 0.3, because they represent noval industries and traditional industries, respectively. And, TSLA show low correlation with BA, Citi, and JPM.

Fig. 1 gives the market prior returns of the assets in a manner of horizontal bar plot, where NVDA has the highest market prior return, and TSLA ranks the second place. However, JPM and GE show relative low market prior returns according to the assumption of Black-Litterman framework.

Three types of investment strategies are built within the proposed GMM-based Black-Litterman theoretical framework: pessimistic-style, optimistic-style, and maximum probability-style. Table V reports the details of posterior returns of the involved portfolios, respectively. It can be observed that the pessimistic-style GMM-BL shows relative high posterior returns comparing to another two strategies, even though the most pessimistic investment views are inputted. The reason for this counter-intuitive phenomenon is that the corresponding confidence level is not high enough to generate significant influence on the prior asset returns. However, the maximum probability portfolio gives the most conservative prior returns due to the highest confidence level.

Table VI reports the details of each portfolio allocation weight, where the optimistic strategy is the most diversified, total 5 stocks are selected by this investment, while the maximum probability model focuses on only 2 assets, MSFT and NVDA. The intelligence of the proposed GMM-BL portfolios can be witnessed by the results of the weight distribution, where the companies with mediocre financial performance such as BA, C and INTC are not considered by any investment strategy.

Table VII presents the evaluation metrics for the proposed portfolios and the equal-weighted portfolio (EW) as the benchmark. ROI and APY gauge the profitability of the portfolios, on which the maximum probability GMM-BL portfolio achieves the highest ROI of 1.3362 and APY of 1.3193. Pessimistic-style GMM-BL portfolio ranks the second place with ROI of 1.1199 and APY of 1.0832. EW does not show any advantage on the profitability, whose ROI and APY are only 0.4660 and 0.4530, respectively.

Regarding to investment maxdrawdown, the optimistic-style GMM-BL portfolio has weak comparative advantage, with MDD of 11.90%. The pessimistic-style GMM-BL shows similar MDD with EW, that is, 12.27% versus 12.52%. The maximum probability GMM-BL has the highest MDD of 14.60%, suggesting the investors may suffer from more potential loss during the fund management period than other strategies. Essentially, MDD provides a clear measure of the worst-case scenario for an investment's decline (downside risk), which is a critical metric for risk-averse investors who want to minimize their exposure to significant losses.

Portfolio volatility (VOL) is quantified by the annual standard deviation, which gives the strategy overall risk level. According to some classical financial theories, investment reaches high return due to it undertakes more risk. Our numerical results also confirm this assertion, where the maximum probability GMM-BL portfolio shows the largest volatility of 35.38%, followed by the pessimistic-style GMM-BL portfolio, 30.63%. Although EW is inferior to other portfolios in terms of ROI and APY, but EW outperform other investments in controlling overall risk level, since its VOL is only 16.77%.

Two risk-adjusted indicators, SR and CR, evaluate portfolio performance comprehensively. SR uses standard deviation as a measure of risk, and assumes the portfolio returns are normally distributed. The maximum probability GMM-BL portfolio dominate other portfolios via its highest SR of 3.6443. The pessimistic-style GMM-BL also has impressive performance with the second highest SR of 3.4382. Nonetheless, the optimistic-style GMM-BL portfolio shows the lowest SR of 2.3, which is even worse

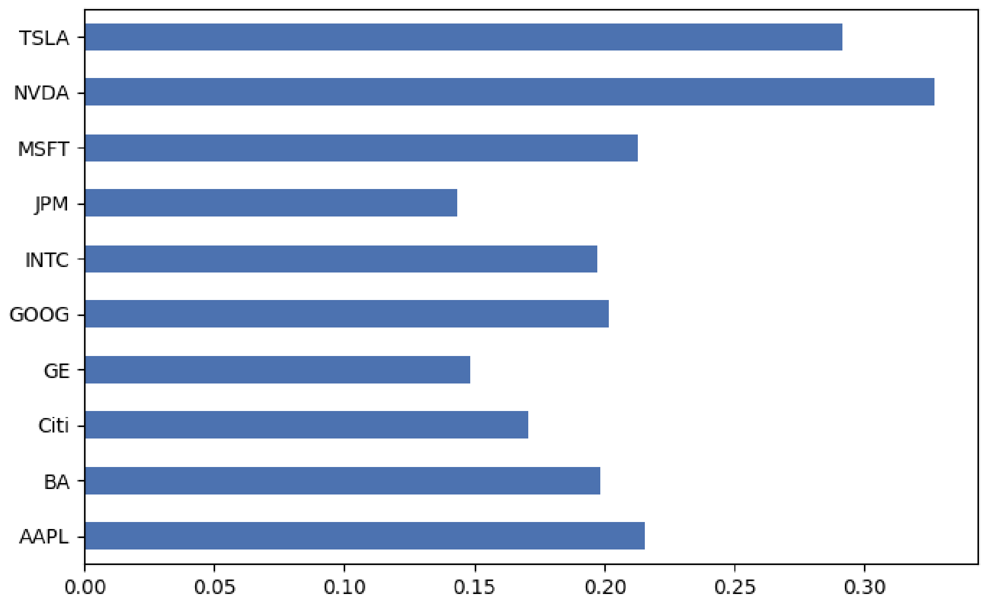

Fig. 1. Market prior returns.

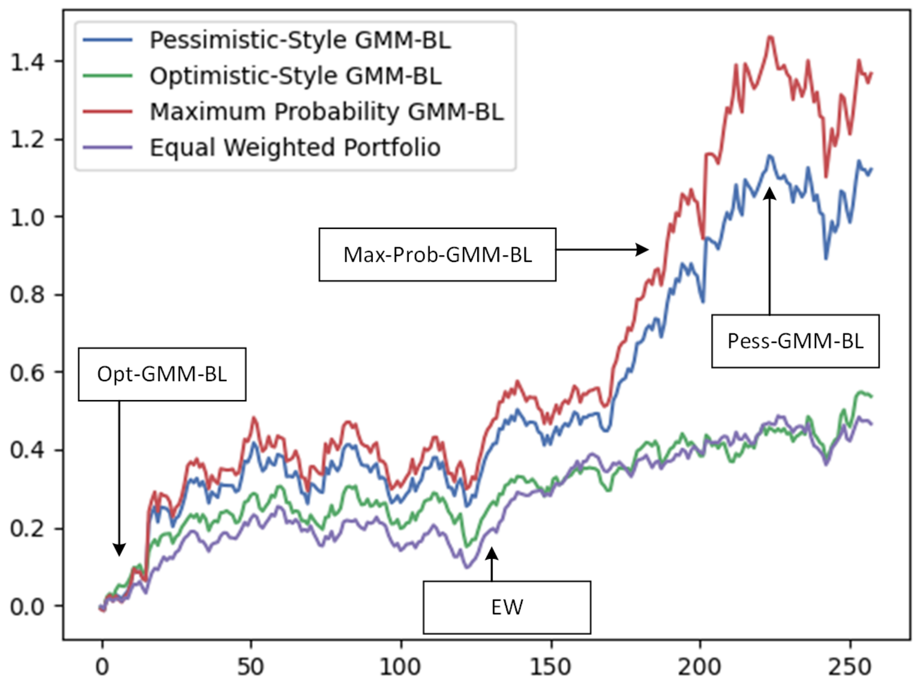

Fig. 2. Portfolio cumulative excess returns.

TABLE IV
RESULTS OF GMM-BL ALGORITHM.

Asset	View1	Confidence1	View2	Confidence2	View3	Confidence3
AAPL	-0.0021	25.54%	0.0037	24.84%	0.0018	49.62%
BA	-0.0012	19.29%	-0.0005	14.53%	0.0004	66.17%
C	0.0003	13.69%	0.0012	39.85%	-0.0002	46.46%
GE	0.0016	18.71%	0.0028	21.83%	0.0006	59.46%
GOOG	-0.0007	23.66%	0.0029	18.11%	0.0013	58.23%
INTC	-0.0032	13.16%	0.0007	54.63%	0.0002	32.21%
JPM	0.0002	15.80%	-0.0004	29.87%	0.0016	54.32%
MSFT	-0.0027	23.36%	0.0026	41.76%	0.0020	34.88%
NVDA	0.0016	40.72%	0.0030	50.65%	0.0010	8.63%
TSLA	-0.0065	54.94%	0.0080	20.66%	0.0200	24.40%

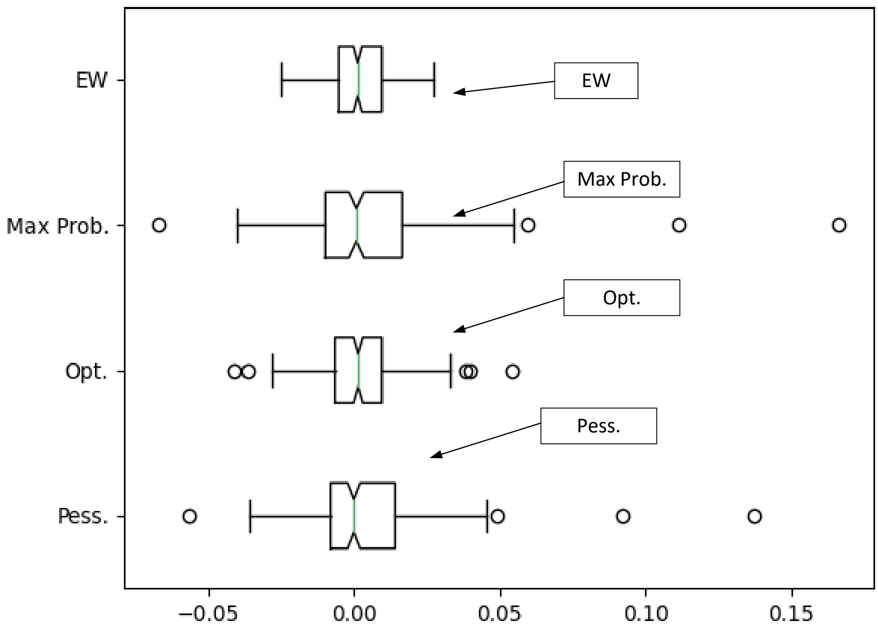

Fig. 3. Portfolio daily returns.

TABLE V
POSTERIOR RETURNS OF THE BUILT PORTFOLIOS.

Asset	Pess. Port.	Opt. Port.	Max Prob. Port.
AAPL	0.0839	0.0656	0.0422
BA	0.0386	0.0174	0.0100
C	0.0309	0.0212	0.0128
GE	0.0220	0.0202	0.0103
GOOG	0.0797	0.0617	0.0350
INTC	0.0694	0.0326	0.0238
JPM	0.0314	0.0203	0.0146
MSFT	0.0872	0.0610	0.0444
NVDA	0.1413	0.0859	0.0660
TSLA	0.0698	0.0837	0.0395

TABLE VI
PORTFOLIO ALLOCATION.

Asset	Pess. Port.	Opt. Port.	Max Prob. Port.
AAPL	10.47%	39.27%	0.00%
BA	0.00%	0.00%	0.00%
C	0.00%	0.00%	0.00%
GE	0.00%	0.00%	0.00%
GOOG	6.82%	30.21%	0.00%
INTC	0.00%	0.00%	0.00%
JPM	0.00%	0.00%	0.00%
MSFT	32.44%	6.25%	37.63%
NVDA	50.27%	17.19%	62.37%
TSLA	0.00%	7.07%	0.00%

TABLE VII
PORTFOLIO ALLOCATION.

Asset	Pess. Port.	Opt. Port.	Max Prob. Port.	EW
ROI	1.1199	0.5367	1.3662	0.4660
APY	1.0832	0.5214	1.3193	0.4530
MDD	12.27%	11.90%	14.60%	12.52%
VOL	30.63%	21.36%	35.38%	16.77%
SR	3.4382	2.3000	3.6443	2.5226
CR	8.8263	4.3818	9.0291	3.6200

than EW.

Slightly different from SR, CR considers the downside risk, MDD, as the risk term. CR is primarily used to evaluate the performance of hedge funds and some other high-volatility investments. As far as this indicator concerned, the maximum probability GMM-BL and the pessimistic-style GMM-BL portfolios continuously maintain their superiority, whose CR values are 9.0291 and 8.8263, respectively. The optimistic-style GMM-BL portfolio surpasses EW in this metric (4.3818 versus 3.62). For the investors with high priority of capital protection, they should pay attention on CR instead of SR.

Fig. 2 visualizes the cumulative excess returns of the involved investment strategies. Except at the very beginning stage, the maximum probability GMM-BL portfolio has higher cumulative excess curves than other portfolios. And, the pessimistic-style shows the sub-optimal cumulative excess curve. However, the two curves are intertwined multiple times during the testing period, indicating they have similar portfolio performances, which is consistent with the results shown in table VII.

Fig. 3 plots the daily returns of the proposed GMM-BL portfolio as well as the baseline strategy. It can be observed that the two portfolios with outstanding profitability, maximum probability GMM-BL and pessimistic-style GMM-BL, have significant outliers, especially on the right side, indicating fascinating potential to reach high level of investment return. The out-of-sample stability of EW model can be reflected on the distribution of daily returns, where almost no outliers can be observed.

F. Sensitivity Analysis

The sensitivity analysis is implemented in this section, where the different numbers of simulated experts are considered. The portfolio performance section mainly reports the results of using 3 defined investment experts corresponding the 3 significant trends could be easily observed from the market index. However, when the candlestick chart is magnified on the finer scale, more sub-trends could be found. Therefore, we consider some more simulated investment experts in forming the proposed three GMM-BL types portfolio models. Note that pick sub-trends is a highly subjective process, thus some human interventions such as parameter tuning are necessary, and the following records of sensitivity analysis are for reference only, because the numerical results would be obviously influenced by different picking rules.

Table VIII gives the sensitivity analysis of portfolio MDD using different numbers of simulated investment experts. For the pessimistic GMM-BL portfolio, it achieves the best MDD performance of 12.20% when the number of simulated experts is 6. The optimistic GMM-BL reaches 11.83% MDD using 7 investment experts. Regarding to the maximum probability portfolio, using 8 simulated experts is a wise choice, since the portfolio MDD is 14.48%.

It also records the sensitivity analysis of portfolio SR with multiple numbers of constructed investment experts. For the pessimistic GMM-BL portfolio, it achieves the best SR performance of 3.4405 when using 6 simulated investment experts. For the optimistic GMM-BL portfolio, using 5 investment experts is appropriate, due to the Sharpe Ratio is 2.3005, higher than other cases. Regarding to the maximum probability GMM-BL portfolio, the best choice is to set 9 investment experts, by which the portfolio SR could reach to 3.6503.

Overall, employing more simulated investment experts in constructing GMM-BL based portfolio does improve the associated performance, but the effectiveness is also limited, especially exceeding a certain threshold. For instance, 5 simulated investment experts in this analysis. Hence, high sensitivity can be observed when the number of simulated experts is small, whereas the sensitivity is low when there exists sufficient number of investment experts.

VI. Conclusions

In this study, we continue to expand and deepen previous researches about the Black-Litterman portfolio [7], [22]. Different from the existing works using machine learning algorithms to generate forecasting investment views, the clustering method is used in this paper, and the achieved centroids are employed as the investment experts for giving representative views. However, this method is slightly conservative, because the Gaussian mixture model could only dig historical samples and give probability-based centroids. Based on that, we construct three types of GMM-BL portfolios on the testing samples, and use $1/N$ strategy as the benchmark for comparison. Empirical study shows that the maximum probability GMM-BL portfolio and the pessimistic style GMM-BL portfolio have impressive performance in terms of profitability. The optimistic-style GMM-BL portfolio show similar performance with EW regarding to both return and risk. Therefore, we can conclude that the GMM can provide reliable investment views for the Black-Litterman portfolio, especially for the short-term portfolio formation.

Some limitations can also be observed from the numerical experiments. Due to both the Black-Litterman theoretical framework and the GMM are based on the Gaussian assumption, whereas the stock returns reject the normality hypothesis to large extent, which generates obvious gap between theory and reality. Even though the GMM-BL portfolios achieve excellent results on the out-of-sample experiments, some obstacles in the real world such as transaction costs and slippage should be taken into account, which construct our future research direction. In addition, we need to consider automating the sensitivity analysis process, where the human interventions should be minimized as much as possible, to ensure the stability of the results.

Acknowledgments

The authors gratefully acknowledge the associated professor from the School of Information Management and Engineering, Shanghai University of Finance & Economics, Jianjun Gao, for his professional guidance. The authors also thank the professor from the School of Information Management and Engineering, Shanghai University of Finance & Economics, Dongmei Han, for her kindly help.

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TABLE VIII
PORTFOLIO PERFORMANCE USING DIFFERENT NUMBER OF SIMULATED EXPERTS.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
Maxdrawdown								
Pess-GMM-BL	0.1277	0.1230	0.1225	0.1220	0.1223	0.1225	0.1222	0.1224
Opt-GMM-BL	0.1190	0.1192	0.1220	0.1188	0.1183	0.1189	0.1191	0.1190
Max-Prob-GMM-BL	0.1460	0.1455	0.1451	0.1450	0.1453	0.1448	0.1449	0.1450
Sharpe Ratio								
Pess-GMM-BL	3.4382	3.4301	3.4384	3.4405	3.4400	3.4397	3.4401	3.4376
Opt-GMM-BL	2.3000	2.2807	2.3101	2.3005	2.2901	2.2900	2.2903	2.2991
Max-Prob-GMM-BL	3.6443	3.6450	3.6501	3.6500	3.6400	3.6408	3.6503	3.6482

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