Soft Topologies Induced by Almost Lower Density Soft Operators

Zanyar A. Ameen* and Ohud F. Alghamdi

Abstract—**In this article, we explore the notion of an "almost lower density soft operator" on an abstract measurable soft space, in addition to its essential properties. Next, we consider the soft topologies generated by these soft operators. We find some criteria ensuring the existence of such soft topologies. We define what it means by the smallest soft topology generated by the intersection of all almost lower density soft operators on a certain abstract measurable soft space. Furthermore, we propose the concept of equivalence of two soft operators on a measurable soft space, and then we analyze the properties of the corresponding soft topologies. We finalize this research by examining several soft topological properties associated with generating soft topologies.**

Index Terms—**soft topology, density soft topology, lower density soft operator, almost lower density soft operator.**

I. Introduction

MANY real-world issues are inherently uncertain, and managing uncertainty well is essential to producing well-informed decisions. The soft set approach is a potent mathematical structure that has gained popularity recently for handling uncertainty and describing the intricate interactions between parameters in a variety of subjects. This approach is especially applicable in areas of difficulty where data could be more precise, clear, and complete, as it provides an adaptable and understandable method of representing and analyzing ambiguous data. Molodtsov [1] expanded on conventional set theory in 1999 through the introduction of the soft set concept. In contrast to crisp sets, which classify components as wholly part of a set or not, soft sets support different levels of ambiguity or uncertainty by allowing for incremental memberships. Because of its adaptability, the soft set framework may be used to address challenges involving uncertainty. This helps decision-makers deal with ambiguous data and express the complex interactions between parameters (see [2], [3]).

Soft topology [4] is a modern version of topology that integrates the classical topology with soft set theory. Soft topology has attracted the interest of many researchers by generalizing numerous types of topological thoughts (see [5], [6], [7], [8], [9], [10]). Methods of generating soft topologies is one of the thoughts. The first non-trivial techniques were introduced in [11] and further developed in [12], [13]. Kandil et al. [14] introduced a method based on the generalized local function that generates soft ideal topologies. In [15], a novel different method for creating soft ideal topologies was demonstrated. The basis of this method was the finding of cluster soft points, functioning as a soft operator of the soft set. Recently, the approach to constructing density soft topologies, which are soft topologies generated by lower density operators (briefly, LDS-operators) on both chargeable and measurable soft spaces, was described in [16], [17]. In this direction, we introduce the concept of an "almost lower density soft operator" (briefly, "ALDS-operator") on measurable soft spaces with the related soft σ-ideal. The aforementioned soft operators are natural extension of the classical lower density and almost lower density operators studied in [18], [19], [20], [21], [22], [23], [24], [25]. Our main goal is to generate soft topologies by ALDS-operators. The generating soft topologies share some properties with density soft topologies. Further features and characterizations of the newly generated soft topologies are discussed. Additionally, the notion of equivalence of two soft operators on a measurable soft space are defined, and some of its implications are explored.

II. Preliminaries

Throughout this note, we refer to our universe as the set $\mathcal{U}$, a set of parameters as $\bar{\xi}$, and any index set as Λ.

Definition II.1. *[1] A soft set over $\mathcal{U}$ is defined to be the pair (S,ξ), where $\xi \subseteq \bar{\xi}$ and $S : \xi \to 2^{\mathcal{U}}$ is a function.*

We call a soft set $\mathcal{U}$ "null" w.r.t. ξ if $S(\varsigma) = \emptyset$ for every $\varsigma \in \xi$ and denote it by $\widetilde{\varnothing}$, and we call it "absolute" w.r.t. ξ if $S(\varsigma) = \mathcal{U}$ for every $\varsigma \in \xi$.

By $\mathbb{S}(\widetilde{\mathcal{U}})$, we mean the collection of all soft subsets over $\mathcal{U}$ related ξ.

Definition II.2. *[26], [27] Let $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$. Then, (S,ξ) is said to be:*

- *finite if for each $\varsigma \in \xi$, $S(\varsigma)$ is finite.*
- *infinite if for some $\varsigma \in \xi$, $S(\varsigma)$ is infinite.*
- *countable if for each $\varsigma \in \xi$, $S(\varsigma)$ is countable.*
- *uncountable if for some $\varsigma \in \xi$, $S(\varsigma)$ is uncountable.*

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Any soft set $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$ can be expanded to $(S,\bar{\xi})$ by associating $S(\varsigma) = \emptyset$ for every $\varsigma \in \bar{\xi} - \xi$.

Let $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$. The complement of (S,ξ) is a soft set $(S^c,\xi) = (S,\xi)^c$, whereas $S^c : \xi \to 2^{\mathcal{U}}$ is a function such that $S^c(\varsigma) = \mathcal{U} - S(\varsigma)$ for all $\varsigma \in \xi$, see [28].

Definition II.3. *[29] Let $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$. Then, (S,ξ) is called a soft point, symbolized with u_ς, if there exists $u \in \mathcal{U}$ and $\varsigma \in \xi$ such that $S(\varsigma) = \{u\}$ and $U(\gamma) = \emptyset$ for all $\gamma \in \bar{\xi} - \{\varsigma\}$. By a statement $u_\varsigma \in (S,\xi)$ we mean $u \in S(\varsigma)$.*

The collection of all soft points in $\mathcal{U}$ w.r.t. ξ is referred to $\mathbb{P}(\widetilde{\mathcal{U}})$.

Definition II.4. *Given $(S_1,\xi) = (S_1,\xi),(S_2,\xi)\widetilde{\subseteq}\mathbb{S}(\widetilde{\mathcal{U}})$. Then (S_1,ξ) is a subset of (S_2,ξ), symbolizes $(S_1,\xi)\widetilde{\subseteq}(S_2,\xi)$, if $S_1(\varsigma) \subseteq S_2(\varsigma)$ for every $\varsigma \in \xi$; (S_i,ξ) is equal to (S_2,ξ), symbolizes $(S_1,\xi) = (S_2,\xi)$, if $(S_1,\xi)\widetilde{\subseteq}(S_2,\xi)$ and $(S_2,\xi)\widetilde{\subseteq}(S_1,\xi)$ for every $\varsigma \in \xi$; The intersection of $(S_1,\xi),(S_2,\xi)$ is the soft set $(S,\xi) = (S_1,\xi)\widetilde{\cap}(S_2,\xi)$ such that $S(\varsigma) = S_1(\varsigma) \cap S_2(\varsigma)$ for every $\varsigma \in \xi$; The union of $(S_1,\xi),(S_2,\xi)$ is the soft set $(S,\xi) = (S_1,\xi)\widetilde{\cup}(S_2,\xi)$ such that $S(\varsigma) = S_1(\varsigma) \cup S_2(\varsigma)$ for every $\varsigma \in \xi$; The set difference between (S_1,ξ) and (S_2,ξ) is the soft set $(S,\xi) = (S_1,\xi) - (S_2,\xi)$ such that $S(\varsigma) = S_1(\varsigma) - S_2(\varsigma)$ for every $\varsigma \in \xi$; The symmetric difference between (S_1,ξ) and (S_2,ξ) is the soft set $(S,\xi) = (S_1,\xi)\widetilde{\Delta}\ (S_2,\xi)$ such that $S(\varsigma) = S_1(\varsigma)\Delta S_2(\varsigma)$ for every $\varsigma \in \xi$, (see [30], [28], [26], [15]). We shall remark that these definitions still hold for any nonempty index set.*

Definition II.5. *Any subclass $\Lambda \subseteq \mathbb{S}(\widetilde{\mathcal{U}})$ that contains $\widetilde{\varnothing}$ and $\widetilde{\mathcal{U}}$ and satisfies the requirement that*

- $(S_1,\xi),(S_2,\xi) \in \Lambda \implies (S_1,\xi)\widetilde{\cap}(S_2,\xi) \in \Lambda$;
- $\{(S_i,\xi) : i \in I\} \in \Lambda \implies \widetilde{\bigcup}_{i\in I}(S_i,\xi) \in \Lambda$,

is considered to be a soft topology over $\mathcal{U}$, see [4], [31]. The triplet $(\mathcal{U},\Lambda,\xi)$ is named a soft topological space. Elements of Λ are called a soft Λ-open sets, or shortly, soft open sets, and their complement are called soft Λ-closed sets or just soft closed sets. We denote the family of all soft closed subsets of $(\mathcal{U},\Lambda,\xi)$ by Λ^c. The lattice of any soft topologies over $\mathcal{U}$ is symbolized by $\mathbb{T}(\widetilde{\mathcal{U}})$, see [32].

Definition II.6. *[4] Let $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$ and $\Lambda \in \mathbb{T}(\widetilde{\mathcal{U}})$. Then:*

1) $Int_\Lambda(S,\xi) = \widetilde{\bigcup}\{(T,\xi) : (T,\xi)\widetilde{\subseteq}(S,\xi),(T,\xi) \in \Lambda\}$ *called the soft interior of (S,ξ).*
2) $Cl_\Lambda(S,\xi) = \widetilde{\bigcap}\{(T,\xi) : (S,\xi)\widetilde{\subseteq}(T,\xi),(T,\xi) \in \Lambda^c\}$ *is called the soft closure of (S,ξ).*

When there is no misunderstanding, we can just utilize $Int(S,\xi)$ and $Cl(S,\xi)$ to represent the soft interior and soft closure of (S,ξ), respectively.

Lemma II.7. *[33] Assume $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$ and $\Lambda \in \mathbb{T}(\widetilde{\mathcal{U}})$. Then,*

$$Int((S,\xi)^c) = (Cl(S,\xi))^c \text{ and } Cl((S,\xi)^c) = (Int(S,\xi))^c.$$

Definition II.8. *Suppose $(S,\xi),(K,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$ and $\Lambda \in \mathbb{T}(\widetilde{\mathcal{U}})$. The soft set (S,ξ) is said to be*

1) *soft nowhere Λ-dense [34] if $Int(Cl(S,\xi)) = \widetilde{\varnothing}$.*
2) *soft Λ-dense in (K,ξ) [35] if $(K,\xi)\widetilde{\subseteq}Cl(S,\xi)$.*
3) *first category soft set [36] if it union of countable soft nowhere Λ-dense sets. Otherwise, it is the second category.*

We may remove Λ from the names of these soft sets if the soft topology is known from the context. The family of soft nowhere Λ-dense sets (resp. first category soft sets) w.r.t. $(\mathcal{U},\Lambda,\xi)$ is named by $\mathbb{N}(\Lambda)$ (resp. $\mathbb{M}(\Lambda)$).

Definition II.9. *[37] Let $(T,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$ and $\Lambda \in \mathbb{T}(\widetilde{\mathcal{U}})$. Then, (T,ξ) is called a soft neighborhood of $u_\varsigma \in \mathbb{P}(\widetilde{\mathcal{U}})$ if there is $(H,\xi) \in \Lambda(u_\varsigma)$ such that $u_\varsigma \in (W,\xi)\widetilde{\subseteq}(T,\xi)$, where $\Lambda(u_\varsigma)$ is the class of soft Λ-open sets containing u_ς.*

Lemma II.10. *[4] For any soft topological space $(\mathcal{U},\Lambda,\xi)$, the set $\Lambda_\varsigma = \{S(\varsigma) : (S,\xi) \in \Lambda\}$ defines a classical topology on $\mathcal{U}$ for every $\varsigma \in \xi$.*

Definition II.11. *[31] For any $(T,\xi) \in \Lambda$, there exists a subset $(M_i,\xi) \in \mathscr{B}$ such that $(T,\xi) = \widetilde{\bigcup}_{i\in I}(M_i,\xi)$. This subset is referred to as a soft base of Λ. If $\mathscr{B}$ is countable, we say Λ has a countable soft base.*

Definition II.12. *[32] Consider $\mathcal{G}\widetilde{\subseteq}\mathbb{S}(\widetilde{\mathcal{U}})$. A soft topology produced by $\mathcal{G}$ refers to the minimal soft topology over $\mathcal{U}$ that contains it.*

Definition II.13. *A soft topological space $(\mathcal{U},\Lambda,\xi)$ is said to be*

1) *soft first countable [10] if every soft point has a countable soft base.*
2) *soft second countable space [10] if it has a countable soft base.*

Definition II.14. *A soft topological space $(\mathcal{U},\Lambda,\xi)$ is soft Lindelöf [10] (resp. soft compact [7]) if each soft open cover of $(\mathcal{U},\Lambda,\xi)$ has a countable (resp. finite) subcover.*

Definition II.15. *[38] A soft topological space $(\mathcal{U},\Lambda,\xi)$ is soft Baire if every no-null soft open set is the second category soft set.*

Definition II.16. *A soft topological space $(\mathcal{U},\Lambda,\xi)$ is said to be*

1) *soft regular [9] if for every $(F,\xi) \in \Lambda^c$ and every $u_\varsigma \notin (F,\xi)$, there are $(S,\xi),(T,\xi) \in \Lambda$ such that $(F,\xi)\widetilde{\subseteq}(S,\xi)$, $u_\varsigma \in (T,\xi)$, and $(S,\xi)\widetilde{\cap}(T,\xi) = \widetilde{\varnothing}$.*
2) *soft T_1 [8] if every $\{u_\varsigma\} \in \Lambda^c$ for every $u_\varsigma \in \mathbb{P}(\widetilde{\mathcal{U}})$.*

Definition II.17. *[14] A class $\widetilde{\varnothing} \neq \mathfrak{I}\widetilde{\subseteq}\mathbb{S}(\widetilde{\mathcal{U}})$ is said to be a soft ideal over $\mathcal{U}$ if $\mathfrak{I}$ has the following properties:*

1) *If $(S,\xi) \in \mathfrak{I}$ and $(K,\xi)\widetilde{\subseteq}(S,\xi)$, then $(K,\xi) \in \mathfrak{I}$, and*
2) *If $(S,\xi),(K,\xi) \in \mathfrak{I}$, then $(S,\xi)\widetilde{\cup}(K,\xi) \in \mathfrak{I}$.*

If (2) is true for a countably large number of soft sets, then $\mathfrak{I}$ is a soft σ-ideal. $\mathbb{I}(\widetilde{\mathcal{U}})$ represents the set of soft σ-ideals over $\mathcal{U}$.

Lemma II.18. *[14] For a soft σ-ideal $\mathfrak{I}$ over $\mathcal{U}$, the set $\mathfrak{I}_\varsigma = \{S(\varsigma) : (S,\xi) \in \mathfrak{I}\}$ defines a σ-ideal on $\mathcal{U}$ for each $\varsigma \in \xi$.*

Definition II.19. *Consider a family $\Sigma \widetilde{\subseteq} \mathbb{S}(\widetilde{\mathcal{U}})$. If Σ includes $\widetilde{\varnothing}$ and is closed under finite (resp. countable) unions and the complement, then Σ is referred to as a soft algebra [39] (resp. soft σ-algebra [40]) over $\mathcal{U}$.*

Lemma II.20. *[41] For a soft σ-algebra Σ over $\mathcal{U}$, the set $\Sigma_\varsigma = \{S(\varsigma) : (S,\xi) \in \Sigma\}$ defines a σ-algebra on $\mathcal{U}$ for every $\varsigma \in \xi$.*

Definition II.21. *[36] Assume $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$, $\Lambda \in \mathbb{T}(\widetilde{\mathcal{U}})$, and $\mathfrak{I} \in \mathbb{I}(\widetilde{\mathcal{U}})$. Then, (S,ξ) is named a soft Λ-open set "modulo" $\mathfrak{I}$ if there exist $(T,\xi) \in \Lambda$ and $(R,\xi) \in \mathfrak{I}$ such that $(S,\xi) = (T,\xi)\widetilde{\Delta}(R,\xi)$. The family of every soft Λ-open set "modulo" $\mathfrak{I}$ is symbolized by $\mathbb{B}_0(\Lambda,\mathfrak{I})$. Whenever $\mathfrak{I}$ is a soft σ-ideal containing the first category soft sets, $\mathbb{B}_0(\Lambda,\mathfrak{I})$ is called a soft σ-algebra of Baire property soft sets.*

Definition II.22. *[36] Suppose $\Lambda \in \mathbb{T}(\widetilde{\mathcal{U}})$. The soft σ-algebra induced by Λ is a Borel soft σ-algebra $\mathbb{B}(\Lambda)$. Objects of $\mathbb{B}(\Lambda)$ are called Borel soft sets.*

Definition II.23. *[42] A point $x \in \mathbb{R}$ is called a Ψ-density point of a Lebesgue measurable set $U \subseteq \mathbb{R}$ if*

$$\lim_{h\to 0^+} \frac{\lambda(U^c \cap [x-h, x+h])}{2h\Psi(2h)} = 0,$$

where Ψ is a continuous nondecreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\lim_{x\to 0^+} \Psi(x) = 0$ and λ is the Lebesgue measure on $\mathbb{R}$.

Definition II.24. *[42] Let $(\mathbb{R},\mathcal{L},\mathcal{N})$ be the Lebesgue measurable space with σ-ideal of sets of measure zero. The family $\Lambda_\Psi = \{U \in \mathcal{L} : U \subseteq \varphi_\Psi(U)\}$ forms a topology on $\mathbb{R}$ called Ψ-density topology generated by the almost lower density operator φ_Ψ, where $\varphi_\Psi(U) = \{x \in \mathbb{R} : x \text{ is a } \Psi\text{-density point of } U\}$.*

The quadruple $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ represents a measurable soft space in the following sense. It consists of Σ, a soft σ-algebra over $\mathcal{U}$, and $\mathfrak{I}$, a soft σ-ideal, such that $\mathfrak{I}\widetilde{\subseteq}\Sigma$ and $\widetilde{\mathcal{U}} \notin \mathfrak{I}$.

Definition II.25. *[16] Let $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ be a measurable soft space. It is said that $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ satisfies the hull property if for each $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$, there exists $(T,\xi) \in \Sigma$ with $(S,\xi)\widetilde{\subseteq}(T,\xi)$ such that for each $(C,\xi) \in \Sigma$ with $(C,\xi)\widetilde{\subseteq}(T,\xi) - (S,\xi)$, we have $(C,\xi) \in \mathfrak{I}$.*

Definition II.26. *[16] Let $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ be a measurable soft space and let $(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}})$. A soft set $(T,\xi) \in \Sigma$ is said to be a measurable kernel of (S,ξ) if $(T,\xi)\widetilde{\subseteq}(S,\xi)$ and for each $(C,\xi)\widetilde{\subseteq}(S,\xi) - (T,\xi)$, we have $(C,\xi) \in \mathfrak{I}$.*

Proposition II.27. *[16] Let $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ be a measurable soft space and let φ_1, φ_2 be LDS-operators on $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. Then $\varphi_1 = \varphi_2$ iff $\Lambda_{\varphi_1} = \Lambda_{\varphi_2}$.*

Theorem II.28. *[16] Assume $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ is a measurable soft space and $\Lambda \in \mathbb{T}(\widetilde{X})$. Then Λ is a soft topology over $\mathcal{U}$ iff $\Sigma = \mathbb{B}_0(\Lambda,\mathfrak{I})$ and $\mathfrak{I} = \mathbb{N}(\Lambda)\widetilde{\cap}\Lambda^c$.*

III. ALDS-Operators

This section introduces the concept of an ALDS-operator on a measurable soft space, followed by some of its consequences.

Definition III.1. *Let $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ be a measurable soft space. The mapping $\varphi : \Sigma \to \mathbb{S}(\widetilde{\mathcal{U}})$ is said to be an almost lower density soft operator on $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$; shortly, an ALDS-operator, if it satisfies the following properties, for each $(S,\xi),(T,\xi) \in \Sigma$:*

(P_1) $\varphi(\widetilde{\mathcal{U}}) = \widetilde{\mathcal{U}}$ *and* $\varphi(\widetilde{\varnothing}) = \widetilde{\varnothing}$*;*
(P_2) $\varphi((S,\xi)\widetilde{\cap}(T,\xi)) = \varphi(S,\xi)\widetilde{\cap}\varphi(T,\xi)$*;*
(P_3) $(S,\xi)\widetilde{\Delta}(T,\xi) \in \mathfrak{I} \Longrightarrow \varphi(S,\xi) = \varphi(T,\xi)$*; and*
(P_4) $\varphi(S,\xi) - (S,\xi) \in \mathfrak{I}$.

If we replace the axiom (P_4) by $(S,\xi)\widetilde{\Delta}\varphi(S,\xi) \in \mathfrak{I}$, soft operator φ is called a lower density soft operator on $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$; in short, an LDS-operator, see [16].

Lemma III.2. *Let $\varphi : \Sigma \to \mathbb{S}(\widetilde{\mathcal{U}})$ be an ALDS-operator on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. Then, for each $\varsigma \in \xi$, $\varphi_\varsigma : \Sigma_\varsigma \to 2^{\mathcal{U}}$ defines an almost lower density operator on a measurable space $(\mathcal{U},\Sigma_\varsigma,\mathfrak{I}_\varsigma)$ (see the definition here [43]).*

Proof: Since φ be an ALDS-operator, then $\varphi(\widetilde{\mathcal{U}}) = \widetilde{\mathcal{U}}$. Therefore, $\varphi(\widetilde{\mathcal{U}}) = \{(\varsigma,\varphi_\varsigma(\mathcal{U})) : \varsigma \in \xi\} = \{(\varsigma,\mathcal{U}) : \varsigma \in \xi\}$. Hence, $\varphi_\varsigma(\mathcal{U}) = \mathcal{U}$ for each ς. Similarly, $\varphi_\varsigma(\emptyset) = \emptyset$ for each ς. Let $S(\varsigma), T(\varsigma) \in \Sigma_\varsigma$. Then $(S,\xi),(T,\xi) \in \Sigma$. By (P_2), we have $\varphi((S,\xi)\widetilde{\cap}(T,\xi)) = \varphi(S,\xi)\widetilde{\cap}\varphi(T,\xi)$. But $\varphi((S,\xi)\widetilde{\cap}(T,\xi)) = \{(\varsigma,\varphi_\varsigma[S(\varsigma)\cap T(\varsigma)]) : \varsigma \in \xi\}$ and $\varphi(S,\xi)\widetilde{\cap}\varphi(T,\xi) = \{(\varsigma,\varphi_\varsigma(S(\varsigma))\cap\varphi_\varsigma(T(\varsigma))) : \varsigma \in \xi\}$. This implies that $\varphi_\varsigma[S(\varsigma)\cap T(\varsigma)] = \varphi_\varsigma(S(\varsigma))\cap\varphi_\varsigma(T(\varsigma))$. Let $S(\varsigma), T(\varsigma) \in \Sigma_\varsigma$ such that $S(\varsigma)\Delta T(\varsigma) \in \mathfrak{I}_\varsigma$ for all $\varsigma \in \xi$. Then $(S,\xi)\widetilde{\Delta}(T,\xi) \in \mathfrak{I}$, which implies $\varphi(S,\xi) = \varphi(T,\xi)$. Thus, $\varphi_\varsigma(S(\varsigma)) = \varphi_\varsigma(T(\varsigma))$. Similarly, we can prove that (P_4) is satisfied for each $\varsigma \in \xi$. This proves φ_ς is an almost lower density operator on $(\mathcal{U},\Sigma_\varsigma,\mathfrak{I}_\varsigma)$. ∎

Lemma III.3. *Let $\varphi : \Sigma \to \mathbb{S}(\widetilde{\mathcal{U}})$ be an ALDS-operator on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ and let $(S,\xi),(T,\xi) \in \Sigma$. Then the following properties hold:*

1) *If $(S,\xi)\widetilde{\subseteq}(T,\xi)$, then $\varphi(S,\xi)\widetilde{\subseteq}\varphi(T,\xi)$.*
2) $\varphi(S,\xi)\widetilde{\cup}\varphi(T,\xi)\widetilde{\subseteq}\varphi[(S,\xi)\widetilde{\cup}(T,\xi)]$.
3) $\mathfrak{I} = \{(S,\xi) \in \Sigma : \varphi(S,\xi) = \widetilde{\varnothing}\}$.

Proof:

1) Let $(S,\xi),(T,\xi) \in \Sigma$ such that $(S,\xi)\widetilde{\subseteq}(T,\xi)$. Then $(S,\xi) = (S,\xi)\widetilde{\cap}(T,\xi)$ and therefore, $\varphi(S,\xi) = \varphi(S,\xi)\widetilde{\cap}\varphi(T,\xi)\ \widetilde{\subseteq}\varphi(T,\xi)$. Hence, $\varphi(S,\xi)\widetilde{\subseteq}\varphi(T,\xi)$.

2) Let $(S,\xi),(T,\xi) \in \Sigma$. Since $(S,\xi)\tilde{\subseteq}(S,\xi)\tilde{\cup}(T,\xi)$ and $(T,\xi)\tilde{\subseteq}(S,\xi)\tilde{\cup}(T,\xi)$, by (1), $\varphi(S,\xi)$ $\tilde{\subseteq}\varphi[(S,\xi)\tilde{\cup}(T,\xi)]$ and $\varphi(T,\xi)\tilde{\subseteq}\varphi[(S,\xi)\tilde{\cup}(T,\xi)]$. Thus, $\varphi(S,\xi)\tilde{\cup}\varphi(T,\xi)\tilde{\subseteq}\varphi[(S,\xi)\tilde{\cup}(T,\xi)]$.
3) Let $(S,\xi) \in \Im$. Since $(S,\xi)\tilde{\Delta}\tilde{\varnothing} \in \Im$, by (P_3), then $\varphi(S,\xi) = \varphi(\tilde{\varnothing}) = \tilde{\varnothing}$. Thus, $\varphi(S,\xi) = \tilde{\varnothing}$.

■

Lemma III.4. *Let φ be soft operator on a measurable soft space $(\mathcal{U},\Sigma,\Im,\xi)$ such that φ satisfies $P_1 - P_3$. Let $\mathcal{S}\tilde{\subseteq}\Sigma$ be a collection such that*

1) *for each $(S,\xi) \in \Sigma$, there exists $(T,\xi) \in \mathcal{S}$ such that $(S,\xi)\tilde{\Delta}(T,\xi) \in \Im$; and*
2) *for each $(S,\xi) \in \mathcal{S}$, $(S,\xi) - \varphi(S,\xi) \in \Im$.*

Then $(S,\xi)\tilde{\Delta}\varphi(S,\xi) \in \Im$ for each $(S,\xi) \in \Sigma$.

Proof: Let $(S,\xi) \in \Sigma$. By the hypotheses, one can conclude that $(S,\xi) - \varphi(S,\xi) \in \Im$. Let $(T,\xi) \in \mathcal{S}$ such that $(S,\xi)^c\tilde{\Delta}(T,\xi) \in \Im$. Therefore, $(S,\xi)\tilde{\cap}(T,\xi) \in \Im$; and thus, $\varphi(S,\xi)\tilde{\cap}\varphi(T,\xi) = \varphi[(S,\xi)\tilde{\cap}(T,\xi)] = \tilde{\varnothing}$. This implies that $\varphi(S,\xi)\tilde{\subseteq}[\varphi(T,\xi)]^c$. Hence, $(S,\xi) - \varphi(S,\xi)\tilde{\Delta}\varphi(S,\xi)$ $\tilde{\cap}(T,\xi)\tilde{\subseteq}(T,\xi) - \varphi(T,\xi) \in \Im$; and so, $(S,\xi) - \varphi(S,\xi) \in \Im$. Consequently, $(S,\xi)\tilde{\Delta}\varphi(S,\xi) \in \Im$. ■

Proposition III.5. *Let $(\mathcal{U},\Sigma,\Im,\xi)$ be a measurable soft space and let $\{\varphi_i : i \in I\}$ be a family of ALDS-operators $\varphi_i : \Sigma \to \mathbb{S}(\tilde{\mathcal{U}})$, then*

$$\varphi = \widetilde{\bigcap}_{i\in I}\varphi_i$$

is an ALDS-operator $(\mathcal{U},\Sigma,\Im,\xi)$.

Proof: Let $(S,\xi),(T,\xi) \in \Sigma$. We shall show the pillars $P_1 - P_4$.

(P_1) Since $\varphi_i(\tilde{\mathcal{U}}) = \tilde{\mathcal{U}}$ and $\varphi_i(\tilde{\varnothing}) = \tilde{\varnothing}$ for all i, then $\varphi(\tilde{\mathcal{U}}) = \widetilde{\bigcap}_{i\in I}\varphi_i(\tilde{\mathcal{U}}) = \tilde{\mathcal{U}}$ and $\varphi(\tilde{\varnothing}) = \widetilde{\bigcap}_{i\in I}\varphi_i(\tilde{\varnothing}) = \tilde{\varnothing}$.

(P_2) For the given soft sets $(S,\xi),(T,\xi) \in \Sigma$, by assumption, we have $\varphi_i((S,\xi)\tilde{\cap}(T,\xi)) = \varphi_i(S,\xi)\tilde{\cap}\varphi_i(T,\xi)$ for all i. Therefore,

$$\begin{aligned}\varphi((S,\xi)\tilde{\cap}(T,\xi)) &= \widetilde{\bigcap}_{i\in I}\varphi_i[(S,\xi)\tilde{\cap}(T,\xi)]\\ &= \widetilde{\bigcap}_{i\in I}[\varphi_i(S,\xi)\tilde{\cap}\varphi_i(T,\xi))]\\ &= \widetilde{\bigcap}_{i\in I}\varphi_i(S,\xi)\tilde{\cap}\widetilde{\bigcap}_{i\in I}\varphi_i(T,\xi))\\ &= \varphi(S,\xi)\tilde{\cap}\varphi(T,\xi).\end{aligned}$$

(P_3) Since, for any $(S,\xi)\tilde{\Delta}(T,\xi) \in \Im$, we have $\varphi_i(S,\xi) = \varphi_i(T,\xi)$ implies $\widetilde{\bigcap}_{i\in I}\varphi_i(S,\xi) = \widetilde{\bigcap}_{i\in I}\varphi_i(T,\xi)$ and so $\varphi(S,\xi) = \varphi(T,\xi)$.

(P_4) By assumption, for all i, $\varphi_i(S,\xi) - (S,\xi) \in \Im$. Therefore $\widetilde{\bigcap}_{i\in I}\varphi_i(S,\xi) - (S,\xi) = \varphi(S,\xi) - (S,\xi) \in \Im$.

Hence, φ is an ALDS-operator $(\mathcal{U},\Sigma,\Im,\xi)$. ■

IV. Soft Topologies Generated by ALDS-Operators

Definition IV.1. *Let φ be an ALDS-operator on a measurable soft space $(\mathcal{U},\Sigma,\Im,\xi)$. If the family*

$$\Lambda_\varphi = \{(S,\xi) \in \Sigma : (S,\xi)\tilde{\subseteq}\varphi(S,\xi)\}$$

is a soft topology over $\mathcal{U}$, then Λ_φ is called the soft topology generated by φ.

Lemma IV.2. *For any ALDS-operator φ on a measurable soft space $(\mathcal{U},\Sigma,\Im,\xi)$. The family Λ_φ can be identified with $\{\varphi(S,\xi) - (T,\xi) : (S,\xi) \in \Sigma, (T,\xi) \in \Im\}$.*

Remark IV.3. *We shall remark that Λ_φ need not be a soft topology, in general, see Example 4.3 in [16]. However, it forms a soft base for a soft topology* (c.f. *[15, Theorem 4.5]).*

Below are some examples of soft topologies generated by ALDS-operators:

Example IV.4. *Let $\mathbb{S}(\tilde{\mathcal{U}})$ be the soft σ-algebra of all soft sets over any nonempty set $\mathcal{U}$ and let $\Im_0 = \{\tilde{\varnothing}\}$ be the soft σ-ideal. The identity soft operator φ on the measurable soft space $(\mathcal{U},\mathbb{S}(\tilde{\mathcal{U}}),\Im_{u_\varsigma},\xi)$ is an ALDS-operator. The family Λ_φ forms the soft discrete topology over $\mathcal{U}$.*

Example IV.5. *Let $(\mathbb{R},\mathbb{B}(\Lambda),\Im_\omega,\xi)$ be a measurable soft space, where $\mathbb{B}(\Lambda)$ is the soft σ-algebra over the set of real numbers $\mathbb{R}$ generated by the natural soft topology Λ (Example 4.3 in [16]) and $\Im_\omega$ is the soft σ-ideal of all countable soft subsets. Consider the soft operator φ on $(\mathbb{R},\mathbb{B}(\Lambda),\Im_\omega,\xi)$ defined by*

$$\varphi(S,\xi) = \begin{cases}\tilde{\varnothing}, & \text{if } (S,\xi)^c \notin \Im_\omega\\ \\ \tilde{\mathbb{R}}, & \text{if } (S,\xi)^c \in \Im_\omega.\end{cases}$$

Then, φ satisfies $P_1 - P_4$ and so it is an ALDS-operator. The family

$$\begin{aligned}\Lambda_\varphi &= \{(S,\xi) \in \mathbb{B}(\Lambda) : (S,\xi)\tilde{\subseteq}\varphi(S,\xi)\}\\ &= \{(S,\xi) \in \mathbb{B}(\Lambda) : (S,\xi)^c \in \Im_\omega\}\tilde{\cup}\{\tilde{\varnothing}\}\end{aligned}$$

is a soft topology over $\mathbb{R}$.

Proposition IV.6. *Let Λ_φ be a soft topology generated by an ALDS-operator φ on a measurable soft space $(\mathcal{U},\Sigma,\Im,\xi)$. For any $(S,\xi) \in \Sigma$, we have*

1) *$Int_{\Lambda_\varphi}(S,\xi)\tilde{\subseteq}(S,\xi)\tilde{\cap}\varphi(S,\xi)$.*
2) *$Cl_{\Lambda_\varphi}(S,\xi)\tilde{\supseteq}(S,\xi)\tilde{\cup}[\varphi((S,\xi)^c)]^c$.*

Proof: Suppose $(S,\xi) \in \Sigma$.

1) By definition, $Int_{\Lambda_\varphi}(S,\xi) = \widetilde{\bigcup}(G_\lambda,\xi)$ such that $(G_\lambda,\xi)\tilde{\subseteq}\varphi(G_\lambda,\xi)$ and $(G_\lambda,\xi)\tilde{\subseteq}(S,\xi)$ for each λ. Since φ is monotone, then we have $(G_\lambda,\xi)\tilde{\subseteq}\varphi(G_\lambda,\xi)\tilde{\subseteq}\varphi(S,\xi)$; and therefore, $(G_\lambda,\xi)\tilde{\subseteq}$ $(S,\xi)\tilde{\cap}\varphi(S,\xi)$ for each λ. Thus, $Int_{\Lambda_\varphi}(S,\xi)\tilde{\subseteq}(S,\xi)\tilde{\cap}\varphi(S,\xi)$.

2) By Remark II.7, we have

$$\begin{aligned} Cl_{\Lambda_\varphi}(S,\xi) &= [Int_{\Lambda_\varphi}[(S,\xi)^c]]^c \\ &\widetilde{\supseteq}[(S,\xi)^c\widetilde{\cap}\varphi((S,\xi)^c)]^c \\ &= (S,\xi)\widetilde{\cup}[\varphi((S,\xi)^c)]^c. \end{aligned}$$

■

Theorem IV.7. *Let Λ_φ be a soft topology generated by an ALDS-operator φ on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. If $(S,\xi)\in\mathfrak{I}$, then $(S,\xi)\in\Lambda_\varphi^c\widetilde{\cap}\mathbb{N}(\Lambda_\varphi)$.*

Proof: Suppose that $(S,\xi)\in\mathfrak{I}$. Obviously, $(S,\xi)^c\in\Sigma$. Since $(S,\xi)^c\widetilde{\Delta}\widetilde{\mathcal{U}}\in\mathfrak{I}$, then $\varphi((S,\xi)^c)=\varphi(\widetilde{\mathcal{U}})$. Therefore, $(S,\xi)^c\widetilde{\subseteq}\widetilde{\mathcal{U}}=\varphi(\widetilde{\mathcal{U}})=\varphi((S,\xi)^c)$, and so, $(S,\xi)^c\in\Lambda_\varphi$. Hence, $(S,\xi)\in\Lambda_\varphi^c$. We now prove that $(S,\xi)\in\mathbb{N}(\Lambda_\varphi)$. If $(G,\xi)\in\Lambda_\varphi$ such that $(G,\xi)\widetilde{\subseteq}(S,\xi)$, then $(G,\xi)\widetilde{\subseteq}\varphi(G,\xi)\widetilde{\subseteq}\varphi(S,\xi)\widetilde{\subseteq}[\widetilde{\mathcal{U}}-\varphi((G,\xi)^c)]=\widetilde{\varnothing}$. This implies that (G,ξ) must be null. Therefore, $(S,\xi)\in\mathbb{N}(\Lambda_\varphi)$. Thus, $(S,\xi)\in\Lambda_\varphi^c\widetilde{\cap}\mathbb{N}(\Lambda_\varphi)$. ■

Corollary IV.8. *Let Λ_φ be a soft topology generated by an ALDS-operator φ on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. Then $\mathfrak{I}\widetilde{\subseteq}\mathbb{M}(\Lambda_\varphi)$.*

The converse of Theorem IV.7 need not always be true, as shown in the following example:

Example IV.9. *Consider the measurable soft space $(\mathbb{R},\mathbb{L}(\mathcal{L}),\mathfrak{I}_\mathcal{N},\xi)$, where $\mathbb{L}(\mathcal{L}),\mathfrak{I}_\mathcal{N}$ are respectively the soft σ-algebra and soft σ-ideal over $\mathbb{R}$ generated by σ-algebra $\mathcal{L}$ of Lebesgue measurable sets and σ-ideal $\mathcal{N}$ of sets of measure zero in $\mathbb{R}$ (Theorem 4.2 in [41]). Let the soft operator $\varphi_{\widetilde{\Psi}}$ on $(\mathbb{R},\mathbb{L}(\mathcal{N}),\mathfrak{I}_\mathcal{N},\xi)$ be defined by $\varphi_{\widetilde{\Psi}}(S,\xi)=\{(\varsigma,\varphi_\Psi(S(\varsigma))):\varsigma\in\xi$ and $S(\varsigma)\in\mathcal{L}\}$. Since φ_Ψ is an almost density operator on $\mathbb{R}$, then $\varphi_{\widetilde{\Psi}}$ is an ALDS-operator. The soft topology $\Lambda_{\varphi_{\widetilde{\Psi}}}$ generated by $\varphi_{\widetilde{\Psi}}$ will be called the $\widetilde{\Psi}$-density soft topology over $\mathbb{R}$. Therefore, $\Lambda_{\varphi_{\widetilde{\Psi}}}$ has many soft $\Lambda_{\widetilde{\Psi}}$-closed and soft $\Lambda_{\widetilde{\Psi}}$-nowhere dense sets that are not in $\mathfrak{I}_\mathcal{N}$, see [42].*

However, the converse is true for LDS-operators on both chargeable and measurable soft spaces, see [17], [16].

Remark IV.10. *In Example IV.9, we shall observe that $\mathbb{M}(\Lambda_{\varphi_{\widetilde{\Psi}}})=\mathbb{S}(\widetilde{\mathbb{R}})$. This proves that a soft topology Λ_φ generated by an ALDS-operator φ need not be soft Baire in contrast to soft topologies generated by LDS-operators, see [17], [16].*

Theorem IV.11. *Let Λ_φ be a soft topology generated by an ALDS-operator φ on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. If $(S,\xi)\in\mathfrak{I}$, then $(S,\xi)\in\Lambda_\varphi^c\widetilde{\cap}\mathbb{D}(\Lambda_\varphi)$, where $\mathbb{D}(\Lambda_\varphi)$ is the family of all soft discrete sets in $(\mathcal{U},\Lambda_\varphi)$. Moreover, the reverse will be true if $\mathfrak{I}$ includes all finite soft sets over $\mathcal{U}$.*

Proof: Suppose $(S,\xi)\in\mathfrak{I}$. By Theorem IV.7, $(S,\xi)\in\Lambda_\varphi^c$. It remains to prove that $(S,\xi)\in\mathbb{D}(\Lambda_\varphi)$. Let u_ς be a soft point in (S,ξ). Since $(S,\xi)-\{u_\varsigma\}\widetilde{\subseteq}(S,\xi)$, then $(S,\xi)-\{u_\varsigma\}\in\mathfrak{I}$. Again, by Theorem IV.7, $(S,\xi)-\{u_\varsigma\}\in\Lambda_\varphi^c$; and hence, $\{u_\varsigma\}$ is a soft Λ_φ-open set in (S,ξ). Evidently, $\{u_\varsigma\}$ is a soft Λ_φ-closed set in (S,ξ). Therefore, $(S,\xi)\in\mathbb{D}(\Lambda_\varphi)$.

Conversely, suppose $(S,\xi)\in\Lambda_\varphi^c\widetilde{\cap}\mathbb{D}(\Lambda_\varphi)$. Clearly, we have that $(S,\xi)^c\in\Lambda_\varphi\widetilde{\subseteq}\Sigma$. Since Σ is a soft σ-algebra, then $(S,\xi)\in\Sigma$. But $(S,\xi)\in\mathbb{D}(\Lambda_\varphi)$; so, for each $u_\varsigma\in(S,\xi)$, there exists $(G_{u_\varsigma},\xi)\in\Lambda_\varphi$ with $u_\varsigma\in(G_{u_\varsigma},\xi)$ such that $(G_{u_\varsigma},\xi)\widetilde{\cap}(S,\xi)=\{u_\varsigma\}$. Therefore,

$$u_\varsigma\in(G_{u_\varsigma},\xi)\widetilde{\subseteq}\varphi(G_{u_\varsigma},\xi)=\varphi[(G_{u_\varsigma},\xi)-\{u_\varsigma\}]\widetilde{\subseteq}\varphi[(S,\xi)^c].$$

This means that $(S,\xi)\widetilde{\subseteq}\varphi[(S,\xi)^c]$; and then, $(S,\xi)=\varphi[(S,\xi)^c]-(S,\xi)^c\in\mathfrak{I}$. ■

Theorem IV.12. *Let Λ_φ be a soft topology generated by an ALDS-operator φ on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. If $\mathfrak{I}=\mathbb{M}(\Lambda_\varphi)$, then $\mathbb{B}_0(\Lambda_\varphi,\mathfrak{I})\widetilde{\subseteq}\Sigma$.*

Proof: Let $(S,\xi)\in\mathbb{B}_0(\Lambda,\mathfrak{I})$. Then $(S,\xi)=(G,\xi)\widetilde{\Delta}(N,\xi)$, where $(G,\xi)\in\Lambda_\varphi$ and $(N,\xi)\in\mathbb{N}(\Lambda_\varphi)$. Since $\Lambda_\varphi\widetilde{\subseteq}\Sigma$, so $(S,\xi)\in\Sigma$. Hence, $\mathbb{B}_0(\Lambda,\mathfrak{I})\widetilde{\subseteq}\Sigma$. ■

Here, we shall remark that for the case of LDS-operators, we have $\mathbb{B}_0(\Lambda_\varphi,\mathfrak{I})=\Sigma$ (see [17], [16]). However, for ALDS-operators, the inclusion is proper, as shown in this example.

Example IV.13. *Consider the measurable soft space $(\mathbb{R},\mathbb{L}(\mathcal{L}),\mathfrak{I}_\mathcal{N},\xi)$ constructed in Example IV.9. Let the soft operator φ on $(\mathbb{R},\mathbb{L}(\mathcal{L}),\mathfrak{I}_\mathcal{N},\xi)$ be defined by*

$$\varphi(S,\xi)=\begin{cases}\widetilde{\varnothing}, & \text{if } (S,\xi)^c\notin\mathfrak{I}_\mathcal{N}\\ \widetilde{\mathbb{R}}, & \text{if } (S,\xi)^c\in\mathfrak{I}_\mathcal{N}.\end{cases}$$

The family

$$\begin{aligned}\Lambda_\varphi &= \{(S,\xi)\in\mathbb{L}(\mathcal{L}):(S,\xi)\widetilde{\subseteq}\varphi(S,\xi)\}\\ &= \{(S,\xi)\in\mathbb{L}(\mathcal{L}):(S,\xi)^c\in\mathfrak{I}_\mathcal{N}\}\widetilde{\cup}\{\widetilde{\varnothing}\}\end{aligned}$$

is a soft topology over $\mathbb{R}$ generated by the ALDS-operator φ such that $\mathfrak{I}_\mathcal{N}=\mathbb{M}(T_\varphi)$ and $\mathbb{B}_0(\Lambda_\varphi,\mathfrak{I}_\mathcal{N})\ \widetilde{\subsetneq}\mathbb{L}(\mathcal{L})$.

Next, we find a condition under which Λ_φ forms a soft topology:

Theorem IV.14. *Let φ be an ALDS-operator on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. If $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ satisfies the hull property, then Λ_φ is a soft topology over $\mathcal{U}$.*

Proof: Suppose $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ satisfies the hull property. We have to prove that Λ_φ is a soft topology. Evidently, we have $\widetilde{\varnothing},\widetilde{\mathcal{U}}\in\Lambda_\varphi$. If $(S,\xi),(T,\xi)\in\Lambda_\varphi$, then $(S,\xi)\widetilde{\subseteq}\varphi(S,\xi)$, $(T,\xi)\widetilde{\subseteq}\varphi(T,\xi)$. Therefore, $(S,\xi)\widetilde{\cap}(T,\xi)\widetilde{\subseteq}\varphi(S,\xi)\widetilde{\cap}\varphi(T,\xi)=\varphi[(S,\xi)\ \widetilde{\cap}(T,\xi)]$ from (P_2); hence, $(S,\xi)\widetilde{\cap}(T,\xi)\in\Lambda_\varphi$. We now check that $\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)\in\Lambda_\varphi$ for each $\{(W_b,\xi):b\in\beta\}\widetilde{\subseteq}\Lambda_\varphi$. Since $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ satisfies the hull property, so there exists a measurable kernel $(C,\xi)\in\Sigma$ of $\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)$ such that $(C,\xi)\widetilde{\subseteq}\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)$ and for any $(D,\xi)\widetilde{\subseteq}\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)-(C,\xi)$ implies $(D,\xi)\in\mathfrak{I}$. Since $[(W_b,\xi)\widetilde{\cap}(C,\xi)]\widetilde{\Delta}(W_b,\xi)\in\mathfrak{I}$ for each $b\in\beta$, then $\varphi[(W_b,\xi)\widetilde{\cap}(C,\xi)]=\varphi(W_b,\xi)$. Now, we obtain that

$$\begin{aligned}(C,\xi)&\widetilde{\subseteq}\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)\widetilde{\subseteq}\widetilde{\bigcup}_{b\in\beta}\varphi(W_b,\xi) \qquad (1)\\ &=\widetilde{\bigcup}_{b\in\beta}\varphi[(W_b,\xi)\widetilde{\cap}(C,\xi)]\widetilde{\subseteq}\varphi(C,\xi).\end{aligned}$$

Since $\varphi(C,\xi)-(C,\xi)\in\mathfrak{I}$, then $\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)-(C,\xi)\in\mathfrak{I}$; and thus, $\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)=[\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)-(C,\xi)]\widetilde{\cup}(C,\xi)\in\Sigma$, as Σ is closed under finite unions. Furthermore, by Equation (1) and Lemma III.3 (1), we have $\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)\widetilde{\subseteq}\varphi(\widetilde{\bigcup}_{b\in\beta}(W_b,\xi))$. Thus, $\widetilde{\bigcup}_{b\in\beta}(W_b,\xi)\in\Lambda_\varphi$. ■

The converse of the above result is not true in general. The family Λ_φ constructed in Example IV.5 by the ALDS-operator φ on the measurable soft space $(\mathbb{R},\mathbb{B}(\Lambda),\mathfrak{I}_\omega,\xi)$ forms a soft topology, while $(\mathbb{R},\mathbb{B}(\Lambda),\mathfrak{I}_\omega,\xi)$ do not satisfy the hull property. It should be noted that the converse is true when φ is an LDS-operator, see Theorem 4.20 in [16].

Theorem IV.15. *Let φ be an ALDS-operator on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ that generates Λ_φ. Then $\mathfrak{I}=\mathbb{M}(\Lambda_\varphi)$ iff there exists a soft σ-algebra $\mathcal{S}\widetilde{\subseteq}\Sigma$ and an LDS-operator θ on $(\mathcal{U},\mathcal{S},\mathfrak{I},\xi)$ such that $\Lambda_\varphi=\Lambda_\theta$.*

Proof: Suppose $\mathfrak{I}=\mathbb{M}(\Lambda_\varphi)$. Choose $\mathcal{S}=\mathbb{B}_0(\Lambda_\varphi,\mathfrak{I})$. Since $\mathbb{B}_0(\Lambda_\varphi,\mathfrak{I})$ is the smallest soft σ-algebra containing Λ_φ and $\mathbb{M}(\Lambda_\varphi)$, so $\mathcal{S}\widetilde{\subseteq}\Sigma$. Set $\theta=\varphi|_{\mathcal{S}}$. We shall check that θ is an LDS-operator θ on a measurable soft space $(\mathcal{U},\mathcal{S},\mathfrak{I},\xi)$. The first three conditions are directly satisfied because φ is an ALDS-operator. We now show that $\theta(S,\xi)\widetilde{\Delta}(S,\xi)\in\mathfrak{I}$. Suppose $(S,\xi)\in\mathcal{S}$. Then $(S,\xi)=(T,\xi)\widetilde{\Delta}(K,\xi)$, where $(T,\xi)\in\Lambda_\varphi$ and $(K,\xi)\in\mathfrak{I}$. It follows that $\theta(S,\xi)=\varphi(S,\xi)=\varphi(T,\xi)\widetilde{\supseteq}(T,\xi)$. Therefore, $(S,\xi)-\theta(S,\xi)\widetilde{\subseteq}(S,\xi)-(T,\xi)\in\mathfrak{I}$; and thus $(S,\xi)-\theta(S,\xi)\in\mathfrak{I}$. By Lemma III.4, we get that $\theta(S,\xi)\widetilde{\Delta}(S,\xi)\in\mathfrak{I}$. Hence, θ is an LDS-operator. The Proposition II.27 guarantees that $\Lambda_\varphi=\Lambda_\theta$.

Conversely, suppose θ is an LDS-operator θ on $(\mathcal{U},\mathcal{S},\mathfrak{I},\xi)$ such that $\Lambda_\varphi=\Lambda_\theta$. By Theorem II.28, $\mathbb{M}(\Lambda_\theta)=\mathfrak{I}$. Since $\Lambda_\varphi=\Lambda_\theta$, then $\mathfrak{I}=\mathbb{M}(\Lambda_\varphi)$. ■

Theorem IV.16. *Let $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ be a measurable soft space satisfying hull property and let $\Lambda\in\mathbb{T}(\widetilde{\mathcal{U}})$ such that $\mathfrak{I},\Lambda\widetilde{\subseteq}\Sigma$. Then there exists a soft σ-algebra Σ^* that makes*

$$\Lambda_{\varphi^*}=\{(S,\xi)\in\Sigma^*:(S,\xi)\widetilde{\subseteq}\varphi^*(S,\xi)\}$$

the smallest soft topology over $\mathcal{U}$ for some ALDS-operator on $(\mathcal{U},\Sigma^*,\mathfrak{I},\xi)$.

Proof: Let $\mathbb{K}$ be the collection of soft σ-algebras containing both $\mathfrak{I}$ and Λ. Set $\Sigma^*=\widetilde{\bigcap}_{\Sigma\in\mathbb{K}}\Sigma$. By Lemma 3.1 in [41], Σ^* is a soft σ-algebra. By assumption and Theorem IV.14, Λ_φ is a soft topology over $\mathcal{U}$ for all ALDS-operators φ. By Proposition III.5, $\varphi^*=\widetilde{\bigcap}\varphi$ is an ALDS-operator. We now show that Λ_{φ^*} is a soft topology. By P_1,P_2 of φ^*, $\widetilde{\varnothing},\widetilde{\mathcal{U}}\in\Lambda_{\varphi^*}$ and $(S,\xi)\widetilde{\cap}(T,\xi)\in\Lambda_{\varphi^*}$ for any $(S,\xi),(T,\xi)$. Let $\{(S_i,\xi):i\in I\}\widetilde{\subseteq}\Lambda_{\varphi^*}$. Then $(S_i,\xi)\widetilde{\subseteq}\varphi^*(S_i,\xi)$ for all i. By monotonicity of φ^*, we can see that $(S_i,\xi)\widetilde{\subseteq}\varphi^*[\widetilde{\bigcup}_{i\in I}(S_i,\xi)]$ and therefore $\widetilde{\bigcup}_{i\in I}(S_i,\xi)\widetilde{\subseteq}\varphi^*[\widetilde{\bigcup}_{i\in I}(S_i,\xi)]$. Thus, $\widetilde{\bigcup}_{i\in I}(S_i,\xi)\in\Lambda_{\varphi^*}$, showing that Λ_{φ^*} is a soft topology over $\mathcal{U}$. The smallestness of Λ_{φ^*} can be followed from the definition of φ^*. ■

Remark IV.17. *From remarks IV.2 and IV.3, we conclude that Λ_{φ^*} identical to the cluster soft topology on a soft σ-algebra generated by some soft topology.*

Definition IV.18. *Let φ_1,φ_2 be soft operators on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. We say φ_1,φ_2 are equivalent, denoted by $\varphi_1\approx\varphi_2$, if $\varphi_1(S,\xi)\widetilde{\Delta}\varphi_2(S,\xi)\in\mathfrak{I}$ for each $(S,\xi)\in\Sigma$.*

Remark IV.19. *We shall remark from Proposition II.27 that φ_1 and φ_2 are equivalent. However, this property does not hold when φ_1 and φ_2 are ALDS-operators, see the below arguments.*

Example IV.20. *Suppose $(\mathbb{R},\mathbb{L}(\mathcal{L}),\mathfrak{I}_\mathcal{N},\xi)$ is the measurable soft space constructed in Example IV.9 and φ is the soft operator on $(\mathbb{R},\mathbb{L}(\mathcal{L}),\mathfrak{I}_\mathcal{N},\xi)$, which is defined by*

$$\varphi_0(S,\xi)=\begin{cases}\varphi_{\Psi}(S,\xi)\widetilde{\cap}(K,\xi), & \text{if } (S,\xi)^c\notin\mathfrak{I}_\mathcal{N}\\ \widetilde{\mathbb{R}}, & \text{if } (S,\xi)^c\in\mathfrak{I}_\mathcal{N},\end{cases}$$

where φ_{Ψ} is the ALDS-operator given in Example IV.9 and $(K,\xi)=\{(\varsigma,K(\varsigma)):\varsigma\in\xi$ and $K(\varsigma)$ is a Bernstein set$\}$. The ALDS-operators φ_0 and φ (defined by Example IV.13) are not equivalent. However, $\Lambda_{\varphi_0}=\Lambda_\varphi$.

On the other hand, one can have the following conclusion for ALDS-operators:

Proposition IV.21. *Let φ_1,φ_2 be ALDS-operators on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. If $\Lambda_{\varphi_1}=\Lambda_{\varphi_2}$, then $\varphi_1(S,\xi)\widetilde{\Delta}\varphi_2(S,\xi)\in\mathfrak{I}$ for each $(S,\xi)\in\Lambda_{\varphi_1}$.*

Proof: Let $\Lambda_{\varphi_1}=\Lambda_{\varphi_2}$. Suppose $(S,\xi)\in\Lambda_{\varphi_1}$. Then $\varphi_1(S,\xi)=(S,\xi)\widetilde{\cup}[\varphi_1(S,\xi)-(S,\xi)]$ and $\varphi_2(S,\xi)=(S,\xi)\widetilde{\cup}[\varphi_2(S,\xi)-(S,\xi)]$. Since $\mathfrak{I}$ is a soft σ-ideal, therefore, $\varphi_1(S,\xi)\widetilde{\Delta}\varphi_2(S,\xi)\in\mathfrak{I}$. ■

Proposition IV.22. *Let φ_1,φ_2 be ALDS-operators on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. If $\varphi_1\approx\varphi_2$, then for each $(S,\xi)\in\Lambda_{\varphi_1}$, there exists $(T,\xi)\in\mathfrak{I}$ such that $(S,\xi)-(T,\xi)\in\Lambda_{\varphi_2}$.*

Proof: Let $(S,\xi)\in\Lambda_{\varphi_1}$. Then $(S,\xi)\widetilde{\subseteq}\varphi_1(S,\xi)$. Set $(T,\xi)=\varphi_1(S,\xi)\widetilde{\Delta}\varphi_2(S,\xi)$. By Proposition IV.21, $(T,\xi)\in\mathfrak{I}$. Therefore, $(S,\xi)\widetilde{\subseteq}\varphi_2(S,\xi)\widetilde{\Delta}(T,\xi)$ and so $(S,\xi)\widetilde{\subseteq}\varphi_2(S,\xi)\widetilde{\cup}(T,\xi)$. This means that $(S,\xi)-(T,\xi)\widetilde{\subseteq}\varphi_2(S,\xi)$. But $\varphi_2(S,\xi)=\varphi_2[(S,\xi)-(T,\xi)]$. Thus,

$(S,\xi)-(T,\xi)\widetilde{\subseteq}\varphi_2[(S,\xi)-(T,\xi)]$. Hence, $(S,\xi)-(T,\xi)\in\Lambda_{\varphi_1}$. ■

Theorem IV.23. *Let φ_1,φ_2 be ALDS-operators on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ and let $\Lambda_{\varphi_1},\Lambda_{\varphi_2}$ be soft topologies generated by φ_1,φ_2, respectively. If $\varphi_1\approx\varphi_2$, then*

1) $\mathbb{M}(\Lambda_{\varphi_1})=\mathbb{M}(\Lambda_{\varphi_2})$*, and*
2) $\mathbb{B}_0(\Lambda_{\varphi_1},\mathbb{M}(\Lambda_{\varphi_1}))=\mathbb{B}_0(\Lambda_{\varphi_2},\mathbb{M}(\Lambda_{\varphi_2}))$.

Proof:

1) To prove this part, it suffices to show that $\mathbb{N}(\Lambda_{\varphi_1})=\mathbb{N}(\Lambda_{\varphi_2})$. Let $(S,\xi)\in\mathbb{N}(\Lambda_{\varphi_1})$ and let $(W,\xi)\in\Lambda_{\varphi_2}$. Then, by Proposition IV.22, there exists $(K,\xi)\in\mathfrak{I}$ such that $(W,\xi)-(K,\xi)\in\Lambda_{\varphi_1}$. Since $(S,\xi)\in\mathbb{N}(\Lambda_{\varphi_1})$, then there exists $\widetilde{\varnothing}\neq(G,\xi)\in\Lambda_{\varphi_1}$ such that $(G,\xi)\widetilde{\subseteq}(W,\xi)-(K,\xi)$ and $(S,\xi)\widetilde{\cap}(G,\xi)=\widetilde{\varnothing}$. Again, by Proposition IV.22, there exists $(L,\xi)\in\mathfrak{I}$ such that $(G,\xi)-(L,\xi)\in\Lambda_{\varphi_2}$. Clearly $\widetilde{\varnothing}\neq(G,\xi)-(L,\xi)\widetilde{\subseteq}(W,\xi)$ and $(S,\xi)\widetilde{\cap}[(G,\xi)-(L,\xi)]=\widetilde{\varnothing}$. Therefore, $(S,\xi)\in\mathbb{N}(\Lambda_{\varphi_2})$; and hence, $\mathbb{N}(\Lambda_{\varphi_1})\widetilde{\subseteq}\mathbb{N}(\Lambda_{\varphi_2})$. The reverse of the inclusion can be proved similarly. Consequently, $\mathbb{M}(\Lambda_{\varphi_1})=\mathbb{M}(\Lambda_{\varphi_2})$.
2) Let $(S,\xi)\in\mathbb{B}_0(\Lambda_{\varphi_1},\mathbb{M}(\Lambda_{\varphi_1}))$. By Theorem 5 (5) in [36], $(S,\xi)=[(G,\xi)-(K,\xi)]\widetilde{\cup}(L,\xi)$ for some $(G,\xi)\in\Lambda_{\varphi_1}$ and $(K,\xi),(L,\xi)\in\mathbb{M}(\Lambda_{\varphi_1})$. By Proposition IV.22, one can find $(W,\xi)\in\mathfrak{I}$ such that $(G,\xi)-(W,\xi)\in\Lambda_{\varphi_2}$. Therefore, $(S,\xi)=[[((G,\xi)-(W,\xi))\widetilde{\cup}((G,\xi)\widetilde{\cap}(W,\xi))]-(K,\xi)]\widetilde{\cup}(L,\xi)=[[(G,\xi)-(W,\xi)-(K,\xi)]\widetilde{\cup}[(G,\xi)\widetilde{\cap}(W,\xi)\widetilde{\cap}(K,\xi)^c]]\widetilde{\cup}(L,\xi)=[[(G,\xi)-(W,\xi)]-(K,\xi)]\widetilde{\cup}[(G,\xi)\widetilde{\cap}(W,\xi)\widetilde{\cap}(K,\xi)^c\widetilde{\cup}(L,\xi)]$. Set $(G',\xi)=(G,\xi)-(W,\xi)$ and $(L',\xi)=(G,\xi)\widetilde{\cap}(W,\xi)\widetilde{\cap}(K,\xi)^c\widetilde{\cup}(L,\xi)$. Since $\mathfrak{I}\widetilde{\subseteq}\mathbb{M}(\Lambda_{\varphi_2})$ and $\mathbb{M}(\Lambda_{\varphi_2})$ is a soft σ-ideal, so $(L',\xi)\in\mathbb{M}(\Lambda_{\varphi_2})$. This means that $(S,\xi)=[(G',\xi)-(K,\xi)]\widetilde{\cup}(L',\xi)$, where $(G',\xi)\in\Lambda_{\varphi_2}$ and $(K,\xi),(L',\xi)\in\mathbb{M}(\Lambda_{\varphi_2})$. Thus, $(S,\xi)\in\mathbb{B}_0(\Lambda_{\varphi_2},\mathbb{M}(\Lambda_{\varphi_2}))$. Consequently, $\mathbb{B}_0(\Lambda_{\varphi_1},\mathbb{M}(\Lambda_{\varphi_1}))\widetilde{\subseteq}\mathbb{B}_0(\Lambda_{\varphi_2},\mathbb{M}(\Lambda_{\varphi_2}))$. By the same technique, we can prove the reverse. Hence, $\mathbb{B}_0(\Lambda_{\varphi_1},\mathbb{M}(\Lambda_{\varphi_1}))=\mathbb{B}_0(\Lambda_{\varphi_2},\mathbb{M}(\Lambda_{\varphi_2}))$.

■

One can conclude from Example IV.20 that the range of an ALDS-operator may be very large. However, it can be determined with Σ.

Theorem IV.24. *Let φ be an ALDS-operator on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$. If there is an ALDS-operator $\Theta:\mathcal{H}\to\Sigma$ on $(\mathcal{U},\mathcal{H},\mathfrak{I},\xi)$, where $\mathcal{H}$ is an $ss\pi$-system such that $\mathfrak{I}\widetilde{\subseteq}\mathcal{H}\widetilde{\subseteq}\Sigma$, then $\Lambda_\Theta=\Lambda_\varphi$.*

Proof: Put $\mathcal{H}=\{(S,\xi)\in\Sigma:\varphi(S,\xi)\in\Sigma\}$. By ($P_1$-$P_2$), $\mathcal{H}$ is an $ss\pi$-system over $\mathcal{U}$ including $\widetilde{\varnothing},\widetilde{\mathcal{U}}$ having the property that $\mathfrak{I}\widetilde{\subseteq}\mathcal{H}\widetilde{\subseteq}\Sigma$. Set $\Theta=\varphi|_{\mathcal{H}}$. Apparently, Θ is an ALDS-operator on $(\mathcal{U},\mathcal{H},\mathfrak{I},\xi)$. To show that $\Lambda_\Theta=\Lambda_\varphi$, it is enough to check that $\Lambda_\varphi\widetilde{\subseteq}\Lambda_\Theta$ since the reverse is always possible. If $(S,\xi)\in\Lambda_\varphi$, then $(S,\xi)\in\Sigma$ and $(S,\xi)\widetilde{\subseteq}\varphi(S,\xi)$. Since $\varphi(S,\xi)\widetilde{\Delta}(S,\xi)\in\mathfrak{I}$, then $\varphi(S,\xi)=(S,\xi)\widetilde{\cup}[\varphi(S,\xi)-(S,\xi)]$; and therefore, $(S,\xi)\in\mathcal{H}$. This means that $(S,\xi)\widetilde{\subseteq}\varphi(S,\xi)=\Theta(S,\xi)$; and hence, $(S,\xi)\in\Lambda_\Theta$. Thus, $\Lambda_\varphi\widetilde{\subseteq}\Lambda_\Theta$. ■

Theorem IV.25. *Let φ be an ALDS-operator on a measurable soft space $(\mathcal{U},\Sigma,\mathfrak{I},\xi)$ and let Λ_φ be a soft topology generated by φ. If $\mathfrak{I}$ contains all finite soft sets, then*

1) $(\mathcal{U},\Lambda_\varphi,\xi)$ *is not soft compact, if $\mathfrak{I}$ contains an infinite soft set.*
2) $(\mathcal{U},\Lambda_\varphi,\xi)$ *is not soft Lindelöf, if $\mathfrak{I}$ contains an uncountable soft set.*
3) $(\mathcal{U},\Lambda_\varphi,\xi)$ *is a soft T_1-space.*
4) $(\mathcal{U},\Lambda_\varphi,\xi)$ *is not soft separable.*
5) $(\mathcal{U},\Lambda_\varphi,\xi)$ *is not soft first countable.*
6) $(\mathcal{U},\Lambda_\varphi,\xi)$ *is not soft second countable.*

Proof:

1) Assume $(S,\xi)\in\mathfrak{I}$ is infinite. For each $u_\varsigma\in\mathbb{P}(\widetilde{\mathcal{U}})$, $(S,\xi)-\{u_\varsigma\}\in\mathfrak{I}$ and so $(S,\xi)-\{u_\varsigma\}\in\Lambda_\varphi^c$. Therefore, $(S,\xi)^c\widetilde{\cup}\{u_\varsigma\}\in\Lambda_\varphi$. This means that $\{(S,\xi)^c\widetilde{\cup}\{u_\varsigma\}\}_{u_\varsigma\in(S,\xi)}$ is a soft Λ_φ-open cover of $\widetilde{\mathcal{U}}$ with no finite subcover. Hence, $(\mathcal{U},\Lambda_\varphi,\xi)$ cannot be soft compact.
2) Similar to (1).
3) The proof is clear since $\{u_\varsigma\}\in\mathfrak{I}$ for each $u_\varsigma\in\mathbb{P}(\widetilde{\mathcal{U}})$ and each element of $\mathfrak{I}$ is soft Λ_φ-closed. Therefore, $(\mathcal{U},\Lambda_\varphi,\xi)$ is soft T_1.
4) Since, by Proposition IV.7, $\mathfrak{I}=\mathbb{N}(\Lambda_\varphi)$, which includes all countable soft sets. Consequently, each countable soft set is soft closed. Then, no countable soft set is soft Λ_φ-dense in $(\mathcal{U},\Lambda_\varphi,\xi)$. Thus, $(\mathcal{U},\Lambda_\varphi,\xi)$ cannot be soft separable.
5) Pick $u_\varsigma\in\mathbb{P}(\widetilde{\mathcal{U}})$ and let $\{(W_n,\xi):n=1,2,\cdots\}$ be a family of soft Λ_φ-open sets containing u_ς. For any n, let $u_\varsigma^n\in(W_n,\xi)$ with $u_\varsigma^n\neq u_\varsigma$. Set $(W,\xi)=(W_1,\xi)-\{u_\varsigma^n:n=1,2,\cdots\}$. Then (W,ξ) is a soft Λ_φ-open set containing u_ς. But, (W,ξ) does not contain an (W_n,ξ) for each n. Therefore, $(\mathcal{U},\Lambda_\varphi,\xi)$ cannot be soft first countable.
6) It follows from (5) since soft second countable space implies soft first countable.

■

Theorem IV.26. *Let $\Lambda\in\mathbb{T}(\widetilde{\mathcal{U}})$ with $\widetilde{\mathcal{U}}\notin\mathbb{M}(\Lambda)$ and let Λ_φ be a soft topology generated by an ALDS-operator φ on the measurable soft space $(\mathcal{U},\mathbb{B}_0(\Lambda,\mathbb{M}(\Lambda)),\mathbb{M}(\Lambda),\xi)$ such that $\Lambda\widetilde{\subseteq}\Lambda_\varphi$. If there exists $(W,\xi)\in\mathbb{M}(\Lambda)$ which soft Λ-dense, then $(\mathcal{U},\Lambda_\varphi)$ is not soft regular.*

Proof: Since φ is defined on the measurable soft space $(\mathcal{U},\mathbb{B}_0(\Lambda,\mathbb{M}(\Lambda)),\mathbb{M}(\Lambda),\xi)$, which satisfied the hull property (see Lemma 4.19 in [16]), by Theorem IV.14, the family

$$\Lambda_\varphi = \{(S,\xi) \in \mathbb{S}(\widetilde{\mathcal{U}}) : (S,\xi) \widetilde{\subseteq} \varphi(S,\xi)\}$$

is a soft topology over $\mathcal{U}$.

Claim: If $(W,\xi) \in \Lambda_\varphi$ is soft Λ-dense, then $(W,\xi)^c \in \mathbb{M}(\Lambda)$.

Proof of the claim. Let $(W,\xi) \in \Lambda_\varphi$ be soft Λ-dense. Then $(W,\xi) = (T,\xi)\widetilde{\Delta}(K,\xi)$ for some $(T,\xi) \in \Lambda$ and $(K,\xi) \in \mathbb{M}(\Lambda)$. We shall show that (T,ξ) is soft Λ-dense. Assume otherwise that there exists $\widetilde{\varnothing} \neq (G,\xi) \in \Lambda$ such that $(T,\xi)\widetilde{\cap}(G,\xi) = \widetilde{\varnothing}$. Since $\Lambda \widetilde{\subseteq} \Lambda_\varphi$, then $(W,\xi)\widetilde{\cap}(G,\xi) \in \Lambda_\varphi$. Therefore, $(W,\xi)\widetilde{\cap}(G,\xi) \widetilde{\subseteq} \varphi(W,\xi)\widetilde{\cap}\varphi(G,\xi) = \varphi((T,\xi)\widetilde{\cap}(G,\xi)) = \widetilde{\varnothing}$. This means that $(W,\xi)\widetilde{\cap}(G,\xi) = \widetilde{\varnothing}$, which contradicts to the soft Λ-density of (W,ξ). Hence, $(T,\xi) \in \Lambda$ which is also soft Λ-dense; and thus, $(T,\xi)^c \in \mathbb{M}(\Lambda)$. Consequently, $(W,\xi)^c \in \mathbb{M}(\Lambda)$.

Suppose $(W,\xi) \in \mathbb{M}(\Lambda)$ is soft Λ-dense and $u_\varsigma \notin (W,\xi)$. If $(\mathcal{U},\Lambda_\varphi)$ is a soft regular space, then there exist $(S,\xi),(T,\xi) \in \Lambda_\varphi$ such that $(W,\xi) \widetilde{\subseteq} (S,\xi)$, $u_\varsigma \in (T,\xi)$, and $(S,\xi)\widetilde{\cap}(T,\xi) = \widetilde{\varnothing}$. By our claim, $(S,\xi)^c \in \mathbb{M}(\Lambda)$ and $(T,\xi)^c \notin \mathbb{M}(\Lambda)$; and thus $(S,\xi)\widetilde{\cap}(T,\xi) \neq \widetilde{\varnothing}$, a contradiction. Hence, $(\mathcal{U},\Lambda_\varphi)$ cannot be soft regular. ∎

V. Conclusion

The term "density topology" is borrowed from the Lebesgue density topology on the set of real numbers. The density topology is larger than the natural (usual) topology. The literature documented a number of density topology extensions. Then an abstract version of density topology was developed by many scholars, see [24], [43], [44], [20], [21]. Recently, the density topology on different domains was introduced in [17], [16]. By weakening the axioms of the lower density operator, an ALDS-operator is born. This soft operator is studied along with its relation to the classical almost lower density operator. Our primary aim is to explore the soft topologies produced by ALDS-operators. Some non-trivial examples of such soft topologies are given. We have compared the properties of soft topologies generated by ALDS-operators to the properties of density soft topologies. In addition, we have defined what it means to be equivalent soft operators. The consequences of this concept are also explored. We also demonstrated which topological properties, like compactness and Lindelofness, some separation axioms, and countability axioms, may be observed in soft topologies formed by ALDS-operators.

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