Degree Based Energies of Amalgamation of Graphs

Swati Nayak, Sabitha D'Souza and Amrithalakshmi*

***Abstract*—Let $G = (V, E)$ be a simple graph of order n and size m, with V as a vertex set and vertex degrees sequence $\Delta = d_1 \geq d_2 \geq \ldots \geq d_n = \delta > 0, d_i = d_G(v_i)$. If the vertices v_i and v_j are adjacent, we denote it as $v_iv_j \in E(G)$. With TI we denote a topological index that can be represented as $TI = TI(G) = \sum_{i\sim j} F(d_i, d_j)$, where F is an appropriately chosen function with the property $F(x, y) = F(y, x)$. A general extended adjacency matrix $A_{TI} = (a_{ij})$ of G is defined as $a_{ij} = F(d_i, d_j)$ if the vertices v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Denote by $f_i, i = 1, 2, ..., n$ the eigenvalues of A_{TI}. The energy of the general extended adjacency matrix is defined as $\varepsilon_{TI} = \varepsilon_{TI}(G) = \sum_{i=1}^{n} |f_i|$. Graph amalgamation is a relationship between two graphs that can provide a way to reduce a graph to a simpler graph while keeping certain structure intact. In this paper, we have obtained various degree based energies of amalgamation of graphs.**

***Index Terms*—Extended Energy, Amalgamation, Extended adjacency matrix.**

I. Introduction

The energy of a graph was first coined by Ivan Gutman. However, the motivation for his work appeared in 1930's. In 1930, German scholar Erich Huckel put forward a method for finding approximate solution of the Schrodinger equation of a class of organic molecules called unsaturated conjugated hydrocarbons. This approach is referred to as Huckel molecular orbital theory(HMO). This is in accordance with the π electron energy of the molecule. Earlier some chemical problems were converted to graph and were then solved using spectral graph theory.

Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges. Let $A = (a_{ij})$ be an adjacency matrix of graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are called eigenvalues of G. The energy of G [1] is defined to be sum of absolute values of the eigenvalues of G. This graph invariant is very closely connected to a chemical quantity known as the total π-electron energy of conjugated hydro carbon molecules. The carbon atoms are represented by the vertices and two vertices are adjacent if and only if there is a carbon-carbon bond. Hydrogen atoms are ignored. The energy level of π-electron in molecules of conjugated hydrocarbons are related to the eigenvalues of a molecular graph. An interesting quantity in Huckel theory is sum of energies of all the electrons in a molecule, so called $\pi-$electron energy of a molecule. For all terminologies we refer [2], [9].

Let $G = (V, E)$ be a simple graph of order n and size m, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, without isolated vertices and sequence of vertex degrees $\delta = d_1 \geq d_2 \geq \ldots \geq d_n = \delta > 0, d_i = d_G(v_i)$. If the vertices v_i and v_j are adjacent we denote it as $v_iv_j \in E(G)$.

Topological indices of graph is defined as

$$TI = TI(G) = \sum_{i\sim j} F(d_i, d_j),$$

where F is a function with the property $F(x, y) = F(y, x)$.

To each topological index TI we can associate a general extended adjacency matrix $A_{TI} = (a_{ij})$ defined as

$$a_{ij} = \begin{cases} F(d_i, d_j), & \text{if } v_iv_j \in E(G) \\ 0 & otherwise. \end{cases}$$

Let $f_1, f_2 \ldots, f_n$ be the eigenvalues of general extended adjacency matrix A_{TI}. Energy of general extended adjacency matrix is defined as $\varepsilon_{TI}(G) = \sum_{i=1}^{n} |f_i|$.

Some well known degree based graph energies are

1) Maximum degree matrix of a graph G [3] is defined as $M(G) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} max(d_i, d_j), & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Maximum degree energy of a graph G is defined as $\varepsilon_M(G) = \sum_{i=1}^{n} |\mu_i|$.

2) Minimum degree matrix of a graph G [4] is defined as $MD(G) = [md_{ij}]$, where

$$md_{ij} = \begin{cases} min(d_i, d_j), & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Minimum degree energy of graph G is given by $\varepsilon_{MD}(G) = \sum_{i=1}^{n} |\xi_i|$.

3) Greatest common divisor matrix of a graph G is defined as $GCD(G) = [g_{ij}]$, where

$$g_{ij} = \begin{cases} g.c.d(d_i, d_j), & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

GCD energy of graph G is $GCD\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$.

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4) First Zagreb matrix $Z^{(1)}$ and second Zagreb matrix $Z^{(2)}$of a graph G [5], respectively defined as

$$(z^{(1)})_{ij} = \begin{cases} d_i + d_j, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

and

$$(z^{(2)})_{ij} = \begin{cases} d_i.d_j, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

First Zagreb energy and second Zagreb energy, respectively defined as

$$Z\varepsilon_1(G) = \sum_{i=1}^{n} |\zeta_i^{(1)}|$$

and

$$Z\varepsilon_2(G) = \sum_{i=1}^{n} |\zeta_i^{(2)}|.$$

5) The Albertson matrix of a graph G [6] is a square matrix $Alb(G) = [a_{ij}]$ of order n with

$$a_{ij} = \begin{cases} |d_i - d_j|, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

The Albertson energy of a graph G is defined as

$$Alb\varepsilon(G) = \sum_{i=1}^{n} |\zeta_i|.$$

6) The harmonic matrix of a graph G is a square matrix $H(G) = [h_{ij}]$ of order n with

$$h_{ij} = \begin{cases} \dfrac{2}{d_i + d_j}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Harmonic energy of a graph G is defined in [7] as

$$H\varepsilon(G) = \sum_{i=1}^{n} |\gamma_i|.$$

7) Geometric-Arithmetic matrix of a graph G is defined as $GA(G) = [(ga)_{ij}]$, where

$$(ga)_{ij} = \begin{cases} \dfrac{2\sqrt{d_id_j}}{d_i + d_j}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Geometric-Arithmetic energy of a graph G is defined in [7] as

$$GA\varepsilon(G) = \sum_{i=1}^{n} |g_i|.$$

8) Arithmetic-Geometric matrix of a graph G is defined as $AG(G) = [(ag)_{ij}]$, where

$$(ag)_{ij} = \begin{cases} \dfrac{d_i + d_j}{2\sqrt{d_id_j}}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Arithmetic-Geometric energy of a graph G is defined in [5] as

$$AG\varepsilon(G) = \sum_{i=1}^{n} |g_i|.$$

9) The sum-connectivity matrix of graph G is $SC(G) = [s_{ij}]$, where

$$s_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i + d_j}}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Sum-connectivity energy of a graph G [7] is defined as

$$SC\varepsilon(G) = \sum_{i=1}^{n} |\mu_i|.$$

10) Randic matrix of graph G is $R(G) = [r_{ij}]$, where

$$r_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i.d_j}}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Randic energy of a graph G [7] is defined as

$$R\varepsilon(G) = \sum_{i=1}^{n} |r_i|.$$

11) Reciprocal Randic matrix of graph G is $RR(G) = [r_{ij}]$, where

$$r_{ij} = \begin{cases} \sqrt{d_i.d_j}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Reciprocal Randic energy of a graph G [5] is defined as

$$RR\varepsilon(G) = \sum_{i=1}^{n} |r_i|.$$

12) Inverse sum indeg matrix of graph G is $ISI(G) = [s_{ij}]$, where

$$s_{ij} = \begin{cases} \frac{d_i.d_j}{d_i + d_j}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Inverse sum indeg energy of a graph G [8] is defined in [7] as

$$ISI\varepsilon(G) = \sum_{i=1}^{n} |\tau_i|.$$

II. General Extended energy of Amalgamation of m copies of complete graph

A graph amalgamation is a relationship between two graphs (one graph is an amalgamation of another). Amalgamations can provide a way to reduce a graph to a simpler graph while keeping certain structure intact.

Definition 2.1: Let $\{G_1, G_2, G_3, \ldots, G_m\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{0i} called a terminal. The amalgamation $Amal(v_{0i}, G_i)$ is formed by taking all the $G_i's$ and identifying their terminals. In particular, if we take $G_i = K_n$ for $i = 1, 2, \ldots, m$ we get amalgamation of m copies of K_n denoted by $Amal(m, K_n), m \geq 2$. For convenience we denote v_0 as the vertex of amalgamation and $v_{j2}, v_{j3}, \ldots, v_{jn}$ are the remaining vertices of the j^{th} copy of K_n, where $1 \leq j \leq m$.

The amalgamation of 3 copies of K_4 is shown in Figure 6.

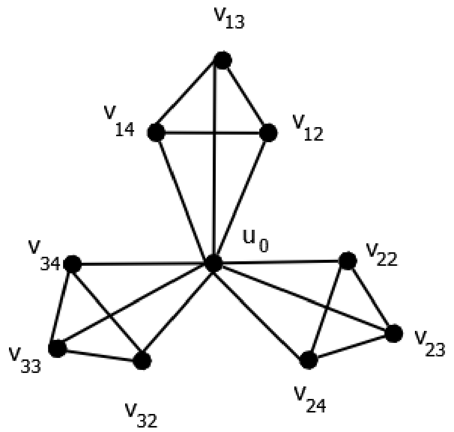

Fig. 1. Amal(3,K_4)

Theorem 2.1: Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$. Then, $\epsilon_{TI}(Amal(m, K_n)) = mc(n-2) + \sqrt{(n-2)^2 + 4ma(n-1)}$.

Proof: Let $A_{TI}(Amal(m, k_n)) =$

$$\begin{pmatrix} 0_1 & aJ_{1\times n-1} & aJ_{1\times n-1} & \ldots & aJ_{1\times n-1} \\ aJ_{n-1\times 1} & cB_{n-1} & 0_{n-1} & \ldots & 0_{n-1} \\ aJ_{n-1\times 1} & 0_{n-1} & cB_{n-1} & \ldots & 0_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ aJ_{n-1\times 1} & 0_{n-1} & 0_{n-1} & \ldots & cB_{n-1} \end{pmatrix}_{m+1}$$

be general extended adjacency matrix of $Amal(m, K_n)$. Here J is matrix of all 1's, 0 is the zero matrix, B is the adjacency matrix of complete sub graph.
Consider $|\lambda I - A_{TI}(Amal(m, k_n))|$.
Applying row operation $R_i \longrightarrow R_i - R_{i+1}, i = 1, 2, \ldots, m$ and column operation $C_i \longrightarrow C_i + C_{i-1} + C_{i-2} + \ldots + C_2, i = m+1, m, \ldots, 3$ on $|\lambda I - A_{TI}(Amal(m, k_n))|$, we get $\lambda^{m-1}(\lambda + c)^{m(n-2)}\{\lambda^2 - c\lambda(n-2) - ma^2(n-1)\}$.
Hence general extended adjacency spectrum of $Amal(m, K_n)$ is

$$\begin{pmatrix} 0 & -c & \frac{P+Q}{2} & \frac{P-Q}{2} \\ m-1 & m(n-2) & 1 & 1 \end{pmatrix},$$

where $P = (n-2)$ and $Q = \sqrt{(n-2)^2 + 4ma(n-1)}$.
So general extended energy of $Amal(m, K_n)$ is
$\varepsilon_{TI}(Amal(m, K_n)) = mc(n-2) + \sqrt{(n-2)^2 + 4ma(n-1)}$. ■

III. General Extended energy of Amalgamation of m copies of Bull graph

Definition 3.1: Bull graph is a planar undirected graph with 5 vertices and 5 edges, in the form of a triangle with two disjoint pendant edges.

Definition 3.2: An amalgamation of bull graph G is a graph formed by joining m copies of bull graph by concatenating the terminal vertex.

Theorem 3.1: Let G be a bull graph. Let $v_0, v_{12}, \ldots, v_{15}, v_{22}, \ldots, v_{25}, \ldots, v_{m1}, v_{m2}, \ldots, v_{m5}$ be the vertices of $Amal(m, G)$. Then,

$$\varepsilon_{TI}(Amal(m, G)) = mc\sqrt{b^2 + 4c^2} + \sqrt{8ma^2 + 4c^2 + b^2}.$$

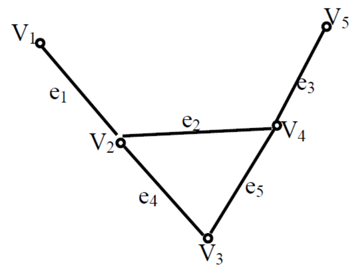

Fig. 2. G: Bull graph

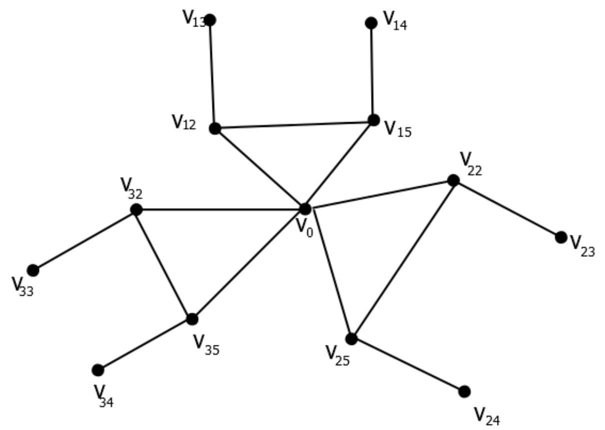

Fig. 3. Amalgamation of bull graph Amal(3,G)

Proof: Let $A_{TI}(Amal(m, G)) =$

$$\begin{pmatrix} 0_1 & A_{1\times 4} & A_{1\times 4} & \ldots & A_{1\times 4} \\ A_{4\times 1} & X_4 & 0_4 & \ldots & 0_4 \\ A_{4\times 1} & 0_4 & X_4 & \ldots & 0_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{4\times 1} & 0_4 & 0_4 & \ldots & X_4 \end{pmatrix}_{m+1}, \text{ where}$$

$A = \begin{pmatrix} a & a & 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & b & c & 0 \\ b & 0 & 0 & c \\ c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}$ be general extended adjacency matrix of $Amal(m, G)$.
Consider $|\lambda I - A_{TI}(Amal(m, G))|$.
Applying row operation $R_i \longrightarrow R_i - R_{i+1}, i = 1, 2, \ldots, m$ and column operation $C_i \longrightarrow C_i + C_{i-1} + C_{i-2} + \ldots + C_2, i = m+1, m, \ldots, 3$ on $|\lambda I - A_{TI}(Amal(m, G))|$, we get

$$(|\lambda I - X|)^{m-1} \begin{vmatrix} \lambda & -Am \\ -A & \lambda I - X \end{vmatrix}_2.$$

Further $|\lambda I - X| = (\lambda^2 - b\lambda - c^2)(\lambda^2 + b\lambda - c^2)$.
And $|\lambda I - A| = \lambda(\lambda^2 + b\lambda - c^2)(\lambda^2 - b\lambda - (2ma^2 + c^2))$.
Hence general extended adjacency spectrum of $Amal(m, G)$ is

$$\begin{pmatrix} 0 & \frac{-P+Q}{2} & \frac{-P-Q}{2} & \frac{P+Q}{2} & \frac{P-Q}{2} & \frac{P+R}{2} & \frac{P-R}{2} \\ 1 & m & m & m-1 & m-1 & 1 & 1 \end{pmatrix},$$

where $P = b$, $Q = \sqrt{b^2 + 4c^2}$ and $R = \sqrt{b^2 + 8ma^2 + 4c^2}$.
So general extended energy of $Amal(m, G)$ is

$$\epsilon_{TI}(Amal(m, G)) = m\sqrt{b^2 + 4c^2} + \sqrt{b^2 + 4c^2 + 8ma^2}.$$

■

TABLE I
GENERAL EXTENDED ENERGY OF AMALGAMATION OF COMPLETE GRAPH K_n.

Sl. No.	Various energies	values of a and c	Energy $\epsilon_{TI}(Amal(m,K_n))$
1	Maximum degree, Minimum degree and gcd energy	$a=c=n-1$	$\varepsilon_M(Amal(m,K_n)) = \varepsilon_{MD}(Amal(m,K_n)) = GCD\varepsilon(Amal(m,K_n)) = m(n-1)(n-2)+\sqrt{(n-2)^2+4m(n-1)^2}$
2	Harmonic energy	$a=\dfrac{1}{n-1}, c=\dfrac{2}{(m+1)(n-1)}$	$H\varepsilon(Amal(m,K_n)) = \dfrac{m(n-2)}{n-1}+\sqrt{(n-2)^2+\dfrac{8m}{m+1}}$
3	Randic energy	$a=\dfrac{1}{\sqrt{m}(n-1)}, c=\dfrac{1}{n-1}$	$R\varepsilon(Amal(m,K_n)) = \dfrac{m(n-2)}{n-1}+\sqrt{(n-2)^2+4\sqrt{m}}$
4	Reciprocal randic energy	$a=\sqrt{m}(n-1), c=n-2$	$RR\varepsilon(Amal(m,K_n)) = m(n-1)(n-2)+\sqrt{(n-2)^2+4m^{3/2}(n-1)^2}$
5	Sum-Connectivity energy	$a=\dfrac{1}{\sqrt{(m+1)(n-1)}}, c=\dfrac{1}{2(n-1)}$	$SC\varepsilon(Amal(m,K_n)) = \dfrac{m(n-2)}{\sqrt{2(n-1)}}+\sqrt{(n-2)^2+4m\sqrt{\dfrac{n-1}{m+1}}}$
6	Geometric-Arithmetic energy	$a=\frac{2\sqrt{m}}{m+1}, c=1$	$GA\varepsilon(Amal(m,K_n)) = m(n-2)+\sqrt{(n-2)^2+\frac{8m^{3/2}(n-1)}{m+1}}$
7	Arithmetic-Geometric energy	$a=\frac{m+1}{2\sqrt{m}}, c=1$	$AG\varepsilon(Amal(m,K_n)) = m(n-2)+\sqrt{(n-2)^2+\frac{4m(n-1)(m+1)}{2\sqrt{m}}}$
8	First Zagreb energy	$a=(m+1)(n-1), c=2(n-1)$	$Z\varepsilon_1(Amal(m,K_n)) = 2m(n-1)(n-2)+\sqrt{(n-2)^2+4m(m+1)(n-1)^2}$
9	Second Zagreb energy	$a=m(n-1)^2, (n-1)^2$	$Z\varepsilon_2(Amal(m,K_n)) = m(n-1^2)(n-2)+\sqrt{(n-2)^2+4m^2(n-1)^3}$
10	Albertson energy	$a=(m-1)(n-1), c=0$	$Alb\varepsilon(Amal(m,K_n)) = \sqrt{(n-2)^2+4m(m-1)(n-1)^2}$
11	Inverse sum indeg energy	$a=\frac{m(n-1)}{m+1}, c=\frac{n-1}{2}$	$ISI\varepsilon(Amal(m,K_n)) = \frac{m(n-1)(n-2)}{2}+\sqrt{(n-2)^2+\frac{4m^2(n-1)^2}{m+1}}$

TABLE II
GENERAL EXTENDED ENERGY OF AMALGAMATION OF BULL GRAPH G.

Sl. No.	Various energies	values of a, b and c	Energy $\epsilon_{TI}(Amal(m,G))$
1	Maximum degree	$a=6, b=c=3$	$\varepsilon_M(Amal(m,G)) = m\sqrt{45}+\sqrt{288m+45}$
2	Minimum degree and gcd energy	$a=b=3, c=1$	$\varepsilon_{MD}(Amal(m,G)) = GCD\varepsilon(Amal(m,G)) = m\sqrt{13}+\sqrt{72m+13}$
3	Harmonic energy	$a=\frac{2}{9}, b=\frac{1}{3}, c=\frac{1}{2}$	$H\varepsilon(Amal(m,G)) = m\sqrt{\frac{10}{9}}+\sqrt{\frac{32m}{81}+\frac{10}{9}}$
4	Randic energy	$a=\frac{1}{3\sqrt{2}}, b=\frac{1}{3}, c=\frac{1}{\sqrt{3}}$	$R\varepsilon(Amal(m,G)) = m\sqrt{\frac{13}{9}}+\sqrt{\frac{4m+13}{9}}$
5	Reciprocal randic energy	$a=3\sqrt{2}, b=3, c=\sqrt{3}$	$RR\varepsilon(Amal(m,G)) = m\sqrt{21}+\sqrt{144m+21}$
6	Sum-Connectivity energy	$a=\frac{1}{3}, b=\frac{1}{\sqrt{6}}, c=\frac{1}{2}$	$SC\varepsilon(Amal(m,G)) = m\sqrt{\frac{7}{6}}+\sqrt{\frac{8m}{9}+\frac{7}{6}}$
7	Geometric-Arithmetic energy	$a=\frac{2\sqrt{2}}{3}, b=1, c=\frac{1}{\sqrt{3}}$	$GA\varepsilon(Amal(m,G)) = m\sqrt{\frac{7}{3}}+\sqrt{\frac{64m}{9}+\frac{7}{3}}$
8	Arithmetic-Geometric energy	$a=\frac{3}{2\sqrt{2}}, b=1, c=\sqrt{3}$	$AG\varepsilon(Amal(m,G)) = m\sqrt{13}+\sqrt{9m+13}$
9	First Zagreb energy	$a=9, b=6, c=4$	$Z\varepsilon_1(Amal(m,G)) = 10m+\sqrt{648m+100}$
10	Second Zagreb energy	$a=18, b=9, c=3$	$Z\varepsilon_2(Amal(m,G)) = m\sqrt{117}+\sqrt{2592m+117}$
11	Albertson energy	$a=3, b=0, c=2$	$Alb\varepsilon(Amal(m,G)) = 4m+\sqrt{72m+16}$
12	Inverse sum indeg energy	$a=2, b=\frac{3}{2}, c=\frac{3}{4}$	$ISI\varepsilon(Amal(m,K_n)) = m\sqrt{\frac{9}{2}}+\sqrt{32m+\frac{9}{2}}$

IV. General Extended energy of Amalgamation of complete graph and star graph

Definition 4.1: The amalgamation of complete graph and star graph is denoted by $Amal(K_n, S_m, v_0)$ is formed by joining K_n and S_n at common vertex v_0.

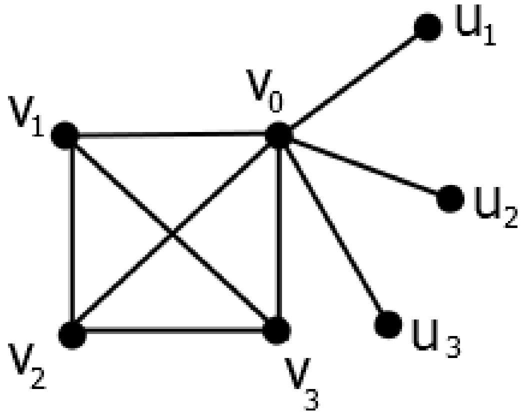

Fig. 4. Amal(K_4, S_3, v_o)

Theorem 4.1: Let $v_1, v_2, \ldots, v_{n-1}, v_0, u_1, u_2 \ldots, u_m$ be the vertices of $Amal(K_n, S_m, v_0)$.
Then $|\lambda I - A_{TI}(Amal(K_n, S_m, v_0))| = \lambda^{m-1}(\lambda + a)^{n-2}$ $[\lambda^3 - (n-2)a\lambda^2 + (b^2(n-1) + c^2)\lambda + c^2(n-2)a]$.

Proof: Let general extended adjacency matrix of $Amal(K_n, S_m, v_0)$ be
$A_{TI}(Amal(K_n, S_m, v_0)) =$

$$\begin{pmatrix} aA(K_{n-1}) & bJ_{n-1\times 1} & 0_{n-1\times m} \\ bJ_{1\times n-1} & 1_1 & cJ_{1\times m} \\ 0_{m\times n-1} & cJ_{m\times 1} & 0_m \end{pmatrix}_{m+n},$$

where J is a matrix of all one's and I is an identity matrix.

Step1: Consider $|\lambda I - A_{TI}(Amal(K_n, S_m, v_0))| =$

$$\begin{vmatrix} \lambda I - aA(K_{n-1}) & -bJ_{n-1\times 1} & 0_{n-1\times m} \\ -bJ_{1\times n-1} & \lambda I_1 & -cJ_{1\times m} \\ 0_{m\times n-1} & -cJ_{m\times 1} & 0_m \end{vmatrix}_{m+n}.$$

Step 2: Applying row operation
$R_i \longrightarrow R_i - R_{i+1}, i = 1, 2, \ldots, n-2, n+1, \ldots, m-1$ on $|\lambda I - A_{TI}(Amal(K_n, S_n, v_0))|$, we get $|B|$.

Step 3: Then employing column operations
$C_i \longrightarrow C_i + C_{i-1} + \ldots + C_{n+1}, i = m, m-1, \ldots, n+1$ and $C_j \longrightarrow C_j + C_{j-1} + \ldots + C_1, i = n-2, n-3, \ldots, 2$ on $|B|$,

we get $\lambda^{m-1}(\lambda + a)^{n-2}|\mathbb{M}|$.
Where

$$|\mathbb{M}| = \begin{vmatrix} \lambda - (n-2)a & -b & 0 \\ -(n-1)b & \lambda & -c \\ 0 & -c & \lambda \end{vmatrix}_3.$$

$$= \lambda^3 - (n-2)a\lambda^2 + (b^2(n-1) + c^2)\lambda + c^2(n-2)a.$$

Hence, general extended characteristic polynomial of $Amal(K_n, S_n, v_0)$ is
$\lambda^{m-1}(\lambda + a)^{n-2}[\lambda^3 - (n-2)a\lambda^2 + (b^2(n-1) + c^2)\lambda + c^2(n-2)a]$. ■

TABLE III
General extended energy of amalgamation of complete graph K_n and star graph S_n

Sl. No.	Various energies	values of a, b and c
1	Maximum degree	$a = n-1$, $b = c = m+n-1$
2	Minimum degree	$a = b = n-1$, $c = 1$
3	gcd energy	$a = 3$, $b = gcd(a, b)$, $c = 1$
4	Randic energy	$a = \frac{1}{n-1}$, $b = \frac{1}{\sqrt{(n-1)(m+n-1)}}$, $c = \frac{1}{\sqrt{m+n-1}}$
5	Reciprocal randic energy	$a = n-1$, $b = \sqrt{(n-1)(m+n-1)}$, $c = \sqrt{m+n-1}$
6	Sum-Connectivity energy	$a = \frac{1}{\sqrt{2(n-1)}}$, $b = \frac{1}{\sqrt{m+2n-2}}$, $c = \frac{1}{m+n}$
7	Geometric-Arithmetic energy	$a = 1$, $b = \frac{2\sqrt{(n-1)(m+n-1)}}{m+2n-2}$, $c = \frac{2\sqrt{m+n-1}}{m+n}$
8	Arithmetic-Geometric energy	$a = 1$, $b = \frac{m+2n-2}{2\sqrt{(n-1)(m+n-1)}}$, $c = \frac{m+n}{2\sqrt{m+n-1}}$
9	First Zagreb energy	$a = 2n-2$, $b = m+2n-2$, $c = m+n$
10	Second Zagreb energy	$a = (n-1)^2$, $b = (n-1)(m+n-1)$, $c = m+n-1$
11	Albertson energy	$a = 0$, $b = m$, $c = m+n-2$
12	Inverse sum indeg energy	$a = \frac{n-1}{2}$, $b = \frac{(n-1)(m+n-1)}{m+2n-2}$, $c = \frac{m+n-1}{m+n}$

V. General Extended energy of Amalgamation of m copies of Fan graph

Definition 5.1: The amalgamation of m copies of Fan graph of order $n+1$ is a denoted by $Amal(F_n, v, m)$ is formed by joining m copies of F_n at common vertex v.

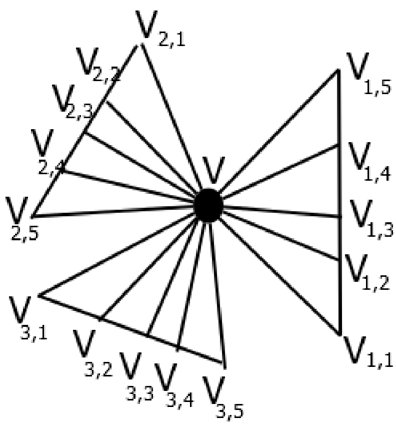

Fig. 5. Amal($F_5, v, 3$)

Theorem 5.1: Let $V = \{v, v_{i,j}/i = 1, 2, \ldots m, j = 1, 2, \ldots 5\}$ be the vertex set of $Amal(F_5, v, m)$. Then $|\lambda I - Amal(F_5, v, m)|$ is $\lambda^{m-1}(\lambda^2 - b^2)^m(\lambda^2 - (b^2 + 2c^2))^{m-1}$

$[\lambda^4-(2ma_1^2+b^2+2c^2+3ma_2^2)\lambda^2-(4cma_2^2+4mba_1a_2)\lambda+4ma_1^2c^2-4mbca_1a_2+mb^2a_2^2]$.

Proof: Let general extended adjacency matrix of $Amal(F_5,v,m)$ be $A_{TI}(Amal(F_5,v,m)) =$

$$\begin{pmatrix} 0_1 & \mathbb{A}_{1\times5} & \mathbb{A}_{1\times5} & \dots & \mathbb{A}_{1\times5} \\ \mathbb{A}_{5\times1} & C_5 & 0_5 & \dots & 0_5 \\ \mathbb{A}_{5\times1} & 0_5 & C_5 & \dots & 0_5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{A}_{5\times1} & 0_5 & 0_5 & \dots & C_5 \end{pmatrix}_{m+1}.$$

Where $C = \begin{pmatrix} 0 & b & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & c & 0 & c & 0 \\ 0 & 0 & c & 0 & b \\ 0 & 0 & 0 & b & 0 \end{pmatrix}$ and

$\mathbb{A} = \begin{pmatrix} a_1 & a_2 & a_2 & a_2 & a_1 \end{pmatrix}$.

Consider $|\lambda I - A_{TI}(Amal(F_5,v,m))|$.

Applying row operation $R_i \longrightarrow R_i - R_{i+1}, i = 2,3\dots,m-1$ and column operation $C_i \longrightarrow C_i + C_{i-1} + \dots + C_2, i = m, m-1, \dots, 3$ on $|\lambda I - A_{TI}(Amal(F_5,v,m))|$,

we get $|\lambda I - C|^{m-1} \begin{vmatrix} \lambda I_1 & -m\mathbb{A}_{1\times5} \\ -\mathbb{A}_{5\times1} & (\lambda I - c)_5 \end{vmatrix}_2$.

Hence, general extended characteristic polynomial of $Amal(F_5,v,m)$ is $\lambda^{m-1}(\lambda^2-b^2)^m(\lambda^2-(b^2+2c^2))^{m-1}$ $[\lambda^4-(2ma_1^2+b^2+2c^2+3ma_2^2)\lambda^2-(4cma_2^2+4mba_1a_2)\lambda+4ma_1^2c^2-4mbca_1a_2+mb^2a_2^2]$. ■

TABLE IV
GENERAL EXTENDED ENERGY OF AMALGAMATION OF FAN GRAPH F_5

Sl. No.	**Various energies**	**values of a_1, a_2, b and c**
1	Maximum degree	$a_1 = a_2 = 5$, $b = c = 3$
2	Minimum degree	$a_1 = b = 2$, $a_2 = c = 3$
3	gcd energy	$a_1 = a_2 = b = 1$, $c = 3$, $c = 1$
4	Randic energy	$a_1 = \frac{1}{10}, a_2 = \frac{1}{15}$, $b = \frac{1}{\sqrt{6}}$, $c = \frac{1}{3}$
5	Reciprocal randic energy	$a_1 = \sqrt{10}, a_2 = \sqrt{15}$, $b = \sqrt{6}$, $c = 3$
6	Sum-Connectivity energy	$a_1 = \frac{1}{7}, a_2 = \frac{1}{\sqrt{8}}$, $b = \frac{1}{\sqrt{5}}$ $c = \frac{1}{8}$
7	Geometric-Arithmetic energy	$a_1 = \frac{2\sqrt{10}}{7}, a_2 = \frac{2\sqrt{15}}{8}$, $b = \frac{2\sqrt{6}}{5}$, $c = 1$
8	Arithmetic-Geometric energy	$a_1 = \frac{7}{2\sqrt{10}}, a_2 = \frac{8}{2\sqrt{15}}$, $b = \frac{5}{2\sqrt{6}}$, $c = 1$
9	First Zagreb energy	$a_1 = 7, a_2 = 8$, $b = 5$, $c = 6$
10	Second Zagreb energy	$a_1 = 10, a_2 = 15$, $b = 6$, $c = 9$
11	Albertson energy	$a_1 = 3, a_2 = 2$, $b = 1$, $c = 0$
12	Inverse sum indeg energy	$a_1 = \frac{10}{7}, a_2 = 15$, $b = \frac{6}{5}$, $c = \frac{3}{2}$
13	Harmonic energy	$a_1 = \frac{2}{7}, a_2 = \frac{1}{4}, b = \frac{2}{5}, c = \frac{1}{3}$

VI. General Extended energy of Amalgamation of m copies of Wheel graph

Definition 6.1: The amalgamation of m copies of Wheel graph of order $n+1$ is a denoted by $Amal(W_n,v,m)$ is formed by joining m copies of W_n at common vertex v.

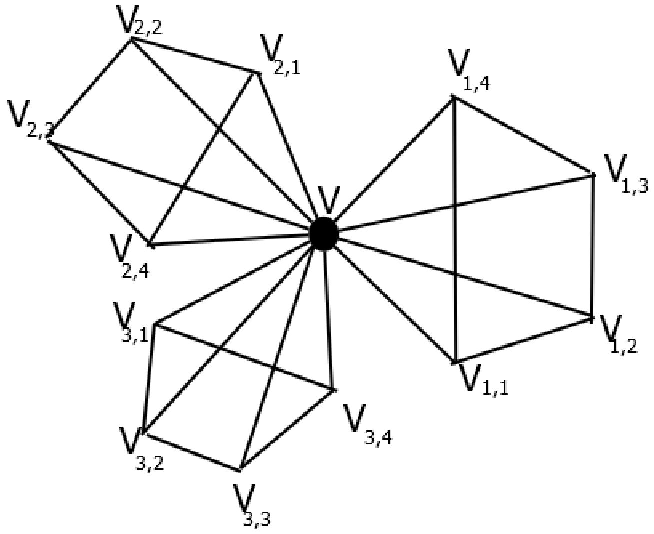

Fig. 6. Amal($W_4, v, 3$)

Theorem 6.1: Let $V = \{v, v_{i,j}/i = 1,2,\dots m, j = 1,2,3,4\}$ be the vertex set of $Amal(W_4,v,m)$. Then

$$\epsilon_{TI}(Amal(W_4,v,m)) = 2(2bm-1+\sqrt{b^2+4a^2m^2}).$$

Proof: Let general extended adjacency matrix of $Amal(W_4,v,m)$ be $A_{TI}(Amal(W_4,v,m)) =$

$$\begin{pmatrix} 0_1 & aJ_{1\times4} & aJ_{1\times4} & \dots & aJ_{1\times4} \\ aJ_{4\times1} & B_4 & 0_4 & \dots & 0_4 \\ aJ_{4\times1} & 0_4 & B_4 & \dots & 0_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ aJ_{4\times1} & 0_4 & 0_4 & \dots & B_4 \end{pmatrix}_{m+1},$$

where $B = \begin{pmatrix} 0 & b & 0 & b \\ b & 0 & b & 0 \\ 0 & b & 0 & b \\ b & 0 & b & 0 \end{pmatrix}$.

Consider $|\lambda I - A_{TI}(Amal(W_4,v,m))| =$

$$\begin{pmatrix} \lambda & -aJ_{1\times4} & -aJ_{1\times4} & \dots & -aJ_{1\times4} \\ -aJ_{4\times1} & \lambda I_4 - B_4 & 0_4 & \dots & 0_4 \\ -aJ_{4\times1} & 0_4 & \lambda I_4 - B_4 & \dots & 0_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -aJ_{4\times1} & 0_4 & 0_4 & \dots & \lambda I_4 - B_4 \end{pmatrix}_{m+1}.$$

Applying row operation $R_i \longrightarrow R_i - R_{i+1}, i = 2,\dots,m-1$ and column operation $C_i \longrightarrow C_i + C_{i-1} + \dots + C_2, i = m, m-1, \dots, 3$ on $|\lambda I - A_{TI}(Amal(W_4,v,m))|$,

we get

$$|\lambda I - B|^{m-1} \begin{vmatrix} \lambda I_1 & -amJ_{1\times4} \\ -aJ_{4\times1} & (\lambda I - B)_4 \end{vmatrix}_2.$$

Hence, general extended spectrum of $Amal(W_4,v,m)$ is

$$\begin{pmatrix} 0 & 2b & -2b & b+\sqrt{b^2+4a^2m^2} & b-\sqrt{b^2+4a^2m^2} \\ 2m & m-1 & m & 1 & 1 \end{pmatrix}.$$

Thus general extended energy of $Amal(W_4,v,m)$ is

$$\epsilon_{TI}(Amal(W_4,v,m)) = 2(2bm-1+\sqrt{b^2+4a^2m^2}).$$

■

TABLE V
GENERAL EXTENDED ENERGY OF AMALGAMATION OF m COPIES OF WHEEL GRAPH G

Sl. No.	**Various energies**	**values of** a, b	**Energy** $\epsilon_{TI}(Amal(W_4, v, m))$
1	Maximum degree	$a = 4, b = 3$	$\varepsilon_M(Amal(W_4, v, m)) = 2(6m - 1 + \sqrt{64m^2 + 9})$
2	Minimum degree energy	$a = b = 3$	$\varepsilon_{MD}(Amal(m, G)) = 2(6m - 1 + \sqrt{36m^2 + 9})$
3	Harmonic energy	$a = \frac{2}{7}, b = \frac{1}{3}$	$H\varepsilon(Amal(W_4, v, m)) = 2\left(\frac{2m}{3} - 1 + \sqrt{\frac{16m^2}{49} + \frac{1}{9}}\right)$
4	Randic energy	$a = \frac{1}{2\sqrt{3}}, b = \frac{1}{3}$	$R\varepsilon(Amal(W_4, v, m)) = 2\left(\frac{2m}{3} - 1 + \sqrt{\frac{m^2}{3} + \frac{1}{9}}\right)$
5	Reciprocal randic energy	$a = 2\sqrt{3}, b = 3$	$RR\varepsilon(Amal(W_4, v, m)) = 2(6m - 1 + \sqrt{48m^2 + 9})$
6	Sum-Connectivity energy	$a = \frac{1}{\sqrt{7}}, b = \frac{1}{\sqrt{6}}$	$SC\varepsilon(Amal(W_4, v, m)) = 2\left(\frac{2m}{\sqrt{6}} - 1 + \sqrt{\frac{4m^2}{7} + \frac{1}{6}}\right)$
7	Geometric-Arithmetic energy	$a = \frac{4\sqrt{3}}{7}, b = 1$	$GA\varepsilon(Amal(W_4, v, m)) = 2\left(2m - 1 + \sqrt{\frac{192m^2}{49} + 1}\right)$
8	Arithmetic-Geometric energy	$a = \frac{7}{4\sqrt{3}}, b = 1$	$AG\varepsilon(Amal(W_4, v, m)) = 2\left(2m - 1 + \sqrt{\frac{49m^2}{12} + 1}\right)$
9	First Zagreb energy	$a = 7, b = 6$	$Z\varepsilon_1(Amal(W_4, v, m)) = 2(12m - 1 + \sqrt{196m^2 + 36})$
10	Second Zagreb energy	$a = 12, b = 9$	$Z\varepsilon_2(Amal(W_4, v, m)) = 2(18m - 1 + \sqrt{576m^2 + 81})$
11	Albertson energy	$a = 1, b = 0$	$Alb\varepsilon(Amal(W_4, v, m)) = 2(2m - 1)$
12	Inverse sum indeg energy	$a = \frac{12}{7}, b = \frac{3}{2}$	$ISI\varepsilon(Amal(W_4, v, m)) = 2\left(3m - 1 + \sqrt{\frac{576m^2}{49} + \frac{9}{4}}\right)$
13	gcd energy	$a = 1, b = 3$	$GCD\varepsilon(Amal(m, G)) = 2(6m - 1 + \sqrt{4m^2 + 9})$

VII. CONCLUDING REMARKS

In this article, we have obtained expression for general extended energy of amalgamation of complete graph, amalgamation of bull graph and amalgamation of wheel graph. Also we obtained general extended characteristic polynomial for amalgamation of complete and star graph and also that of fan graph. We have also compared various degree based energies of $Amal(m, K_n)$, $Amal(W_4, m, v)$ and $Amal(m, G)$.

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