Energy of Partial Complement of a Graph with Self-loops

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***Abstract*—The purpose of this paper is to extend the concept of energy of a graph with self-loops to partial complement of a graph. The partial complement of a graph G with respect to a set S denoted by $G \oplus S$ is the graph obtained by removing the edges of $\langle S \rangle$ and adding edges which are not in $\langle S \rangle$ in G. Let G be a graph of order n with σ self-loops, then the energy is defined as $E(G) = \sum_{i=1}^{n} \left|\lambda_i - \frac{\sigma}{n}\right|$. Based on this, we introduce the concept of energy of partial complement of a graph G with self-loops.**

***Index Terms*—Partial complement, Energy, Self-loop.**

I. Introduction

Let $G = (V, E)$ be a simple, undirected graph of order n and size m. In 1978, I.Gutman defined energy of the graph by defining the adjacency matrix $A(G)$ as,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent the zeros of characteristic polynomial of $A(G)$ [5].
Let G_S be the graph with self-loops to each vertex belonging to S, where $S \subseteq V(G)$. Then the adjacency matrix of G_S is a symmetric square matrix $A(G_S)$ of order n such that,

$$A(G_S)_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent,} \\ 1, & \text{if } i = j \text{ and } v_i \in S, \\ 0, & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

The energy of G_S is defined as,

$$E(G_S) = \sum_{i=1}^{n} \left|\lambda_i(G_S) - \frac{\sigma}{n}\right|,$$

where σ is the cardinality of S and $\lambda_1(G_S), \lambda_2(G_S) \ldots,$ $\lambda_n(G_S)$ are the eigenvalues of $A(G_S)$ [8].

Let $G = (V, E)$ be a graph and $S \subseteq V$. The partial complement of a graph G with respect to S, denoted by $G \oplus S$, is a graph (V, E_S), where for any two vertices $u, v \in V$, $uv \in E_S$ if and only if one of the following conditions hold good [3]:

1) $u \notin S$ or $v \notin S$ and $uv \in E$.
2) $u, v \in S$ and $uv \notin E$.

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Let $(G \oplus S)_r$ be the graph obtained by attaching a self-loop to each of its vertices belonging to $\langle V - S \rangle$ such that $|V - S| = r$. Then the adjacency matrix of $(G \oplus S)_r$ is a symmetric square matrix $A(G \oplus S)_r$ of order n whose $(i, j)-$ entries are

$$A((G \oplus S)_r)_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 1, & \text{if } i = j \text{ and } v_i \in \text{ (V - S)}, \\ 0, & \text{Otherwise.} \end{cases}$$

Let $\chi_1, \chi_2, \ldots, \chi_n$ be the zeros of the characteristic polynomial of $(G \oplus S)_r$ and $\sum_{i=1}^{n} \chi_i = r$.
Then, the energy of $(G \oplus S)_r$ is defined as
$E(G \oplus S)_r = \sum_{i=1}^{n} \left|\chi_i - \frac{r}{n}\right|.$

Example 1.1: For the graph G, $G \oplus S$ and $(G \oplus S)_r$ is as follows.

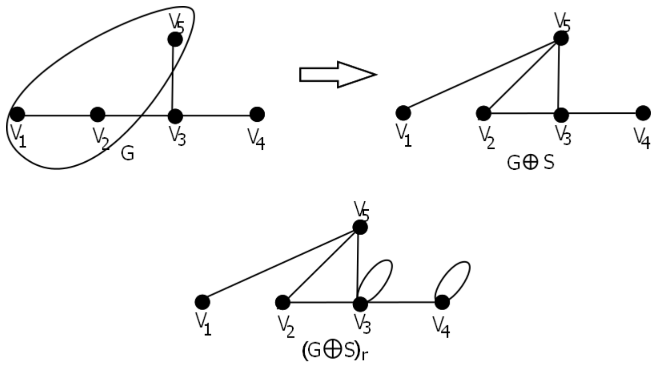

Fig. 1

And the adjacency matrix of partial complement of a graph with self-loop is given by,

$$A(G \oplus S)_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Theorem 1.1: The eigenvalues $\chi_1, \chi_2, \ldots, \chi_n$ of partial complementary graph with self-loops satisfies the following relations:

1) $\sum_{i=1}^{n} \chi_i = r.$
2) $\sum_{i=1}^{n} \chi_i^2 = r + 2m_S$, where m_S be the number of edges of $(G \oplus S)_r$.

Proof:

1) We know that sum of eigenvalues of $A(G \oplus S)_r$=trace of $A(G \oplus S)_r = r$.

2) We know that sum of squares of eigenvalues of $A(G\oplus S)_r$ is trace of $A^2(G\oplus S)_r$.

$$\begin{aligned}\sum_{i=1}^{n}\chi_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}\\ &= \sum_{i=1}^{n}a_{ii}^2+\sum_{i\neq j}a_{ij}a_{ji}\\ &= \sum_{i=1}^{n}a_{ii}^2+2\sum_{i<j}a_{ij}^2\\ &= r+2[m_S].\end{aligned}$$

■

Theorem 1.2: With the same notation as in theorem 1.1, we have $\sum_{i=1}^{n}\left|\chi_i-\frac{r}{n}\right|^2=r+2m_S-\frac{r^2}{n}$, where m_S is the number of edges of $(G\oplus S)_r$.

Proof:

$$\begin{aligned}\sum_{i=1}^{n}\left(\chi_i-\frac{r}{n}\right)^2 &= \sum_{i=1}^{n}\chi_i^2+\frac{r^2}{n}-\frac{2r}{n}\sum_{i=1}^{n}\chi_i\\ &= r+2m_S-\frac{r^2}{n}.\end{aligned}$$

■

Lemma 1.3: 1) If $r=0$, then $E(G\oplus S)_r=E(\overline{G})$.
2) If $r=n$, then $E(G\oplus S)_r=E(G)$.

Proof:

1) If $r=0$, then $|S|=n$ and the graph $(G\oplus S)_r$ coincides with $\overline{G}$. So, $E(G\oplus S)_r=E(\overline{G})$.
2) If $r=n$, then $|S|=0$ and $A(G\oplus S)_r=A(G)+I_n$ where I_n is an identity matrix. Therefore, $\chi(G\oplus S)_r=\chi(G)+1$.
From (I) and (I), $E(G\oplus S)_r=E(G)$.

■

II. Bounds for energy of partial complement of graphs with self-loops

In this section, we discuss the bounds for energy of partial complement of graphs with self-loops.

Theorem 2.1: If $(G\oplus S)_r$ is partial complementary graph with r self-loops then,

$$\sqrt{2m_S+r-\frac{r^2}{n}+n(n-1)\left[det\left(A(G\oplus S)_r-\frac{r}{n}\right)\right]^{2/n}}\le E(G\oplus S)_r\le\sqrt{n\left(r+2m_S-\frac{r^2}{n}\right)}.$$

Proof: By taking $a_i=1$ and $b_i=\left|\chi_i-\frac{r}{n}\right|$ in Cauchy-Schwarz inequality, we get

$$\left(\sum_{i=1}^{n}\left|\chi_i-\frac{r}{n}\right|\right)^2\le n\sum_{i=1}^{n}\left|\chi_i-\frac{r}{n}\right|^2$$

$$\left(\sum_{i=1}^{n}\left|\chi_i-\frac{r}{n}\right|\right)^2\le n\left(r+2m_S-\frac{r^2}{n}\right)$$

$$E(G\oplus S)_r\le\sqrt{n\left(r+2m_S-\frac{r^2}{n}\right)}.$$

By Arithmetic and Geometric mean inequality,

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}\left|\chi_i-\frac{r}{n}\right|\left|\chi_j-\frac{r}{n}\right| &\ge \left[\prod_{i\neq j}\left|\chi_i-\frac{r}{n}\right|\left|\chi_j-\frac{r}{n}\right|\right]^{\frac{1}{n(n-1)}}\\ &\ge\left[\det\left(A(G\oplus S)_r-\frac{r}{n}I\right)\right]^{2/n}\end{aligned}$$

$$\sum_{i\neq j}\left|\chi_i-\frac{r}{n}\right|\left|\chi_j-\frac{r}{n}\right|\ge n(n-1)\left[\det\left(A(G\oplus S)_r-\frac{r}{n}\right)\right]^{2/n}.$$

Consider,

$$\begin{aligned}[E(G\oplus S)_r]^2 &= \left(\sum_{i=1}^{n}\left|\chi_i-\frac{r}{n}\right|\right)^2\\ &= \sum_{i=1}^{n}\left|\chi_i-\frac{r}{n}\right|^2+\sum_{i\neq j}\left|\chi_i-\frac{r}{n}\right|\left|\chi_j-\frac{r}{n}\right|.\end{aligned}$$

$$E(G\oplus S)_r\ge\sqrt{2m_S+r-\frac{r^2}{n}+n(n-1)\left[det\left|A(G\oplus S)_r-\frac{r}{n}\right|\right]^{2/n}}.$$

■

Theorem 2.2: Let $\rho(G\oplus S)_r$ be the spectral radius of $A(G\oplus S)_r$ of order n and r self-loops. Then $\sqrt{\frac{r+2m_S}{n}}\le\rho(G\oplus S)_r\le\sqrt{r+2m_S}$.

Proof: Consider,

$$\begin{aligned}\rho^2(G\oplus S)_r &= \max_{1\le i\le n}\{|\chi_i|^2\}\\ &\le\sum_{i=1}^{n}\chi_i^2=r+2m_S.\end{aligned}$$

$$\rho(G\oplus S)_r\le\sqrt{r+2m_S}.$$

Next consider,

$$\begin{aligned}n\rho^2(G\oplus S)_r &\ge \max_{1\le i\le n}\{|\chi_i|^2\}\\ &\ge r+2m_S.\end{aligned}$$

Thus,

$$\rho(G\oplus S)_r\ge\sqrt{\frac{r+2m_S}{n}}.$$

Hence, $\sqrt{\frac{r+2m_S}{n}}\le\rho(G\oplus S)_r\le\sqrt{r+2m_S}$. ■

Theorem 2.3: If $\chi_1\ge\chi_2\ge\ldots\ge\chi_n$ are the eigenvalues of $A(G\oplus S)_r$ on n vertices, m_S edges with r self-loops then

$$E(G\oplus S)_r\le\chi_1-\frac{r}{n}+\sqrt{(n-1)\left(r+2\left(m_S+\frac{r\chi_1}{n}\right)-\frac{r^2(n+1)}{n^2}-\chi_1^2\right)}.$$

Proof: Applying Cauchy Schwarz inequality for $(n-1)$ terms,

$$\left(\sum_{i=2}^{n}\left|\chi_i-\frac{r}{n}\right|\right)^2\le\left(\sum_{i=2}^{n}1\right)\left(\sum_{i=2}^{n}\left(\chi_i-\frac{r}{n}\right)^2\right)$$

$$\left[E(G\oplus S)_r-\left(\chi_1-\frac{r}{n}\right)\right]^2\le(n-1)\left(r+2m_S-\frac{r^2}{n}-\left(\chi_1-\frac{r}{n}\right)^2\right)$$

$$E(G\oplus S)_r\le\chi_1-\frac{r}{n}+\sqrt{(n-1)\left(r+2\left(m_S+\frac{r\chi_1}{n}\right)-\frac{r^2(n+1)}{n^2}-\chi_1^2\right)}.$$

■

Lemma 2.4: [6] Let $x_1,x_2,\ldots,x_n$ and $y_1,y_2,\ldots,y_n$ be real numbers. If there exist real constants x,y,X and Y such that for each $i, i = 1, 2, \ldots, n, x \leq x_i \leq X$ and $y \leq y_i \leq Y$, then

$$\left| n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \right| \leq \alpha(n)(X-x)(Y-y),$$

where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$. Equality holds if and only if $x_1 = x_2 = \ldots = x_n$ and $y_1 = y_2 = \ldots = y_n$.

Theorem 2.5: Let $\chi_1, \chi_2, \ldots, \chi_n$ be the eigenvalues of the graph $(G \oplus S)_r$, containing r self-loops. Then, $E(G\oplus S)_r \geq n\sqrt{\frac{r}{n} + \frac{2m_s}{n^2} - \left(\frac{r}{n}\right)^2 - \frac{1}{4}\left(\left|\chi_1 - \frac{r}{n}\right| - \left|\chi_n - \frac{r}{n}\right|\right)^2}$.

Proof: Let $\left|\chi_1 - \frac{r}{n}\right| \geq \left|\chi_2 - \frac{r}{n}\right| \geq \ldots \geq \left|\chi_n - \frac{r}{n}\right|$. By substituting $x_i = \left|\chi_i - \frac{r}{n}\right|$, $y_i = \left|\chi_i - \frac{r}{n}\right|$, $x = y = \left|\chi_n - \frac{r}{n}\right|$ and $X = Y = \left|\chi_1 - \frac{r}{n}\right|$ and $\alpha(n) \leq \frac{n^2}{4}$ in 2.4, we obtain

$$\left| n\sum_{i=1}^{n}\left|\chi_i - \frac{r}{n}\right|^2 - \left(\sum_{i=1}^{n}\left(\chi_i - \frac{r}{n}\right)\right)^2 \right| \leq \frac{n^2}{4}\left(\left|\chi_1 - \frac{r}{n}\right| - \left|\chi_n - \frac{r}{n}\right|\right)^2.$$

But, $\sum_{i=1}^{n}\left|\chi_i - \frac{r}{n}\right|^2 = r + 2m_S - \frac{r^2}{n}$ and $\sum_{i=1}^{n}\left|\chi_i - \frac{r}{n}\right| = E(G\oplus S)_r$.

Then the above inequality becomes,

$$n\left(r + 2m_S - \frac{r^2}{n}\right) - (E(G\oplus S)_r)^2 \leq \frac{n^2}{4}\left(\left|\chi_1 - \frac{r}{n}\right| - \left|\chi_n - \frac{r}{n}\right|\right)^2.$$

Thus

$$E(G\oplus S)_r \geq n\sqrt{\frac{r}{n} + \frac{2m_s}{n^2} - \left(\frac{r}{n}\right)^2 - \frac{1}{4}\left(\left|\chi_1 - \frac{r}{n}\right| - \left|\chi_n - \frac{r}{n}\right|\right)^2}.$$

■

Lemma 2.6: [6] Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. If there exist real constants r and R such that for each $i, i = 1, 2, \ldots, n, rx_i \leq y_i \leq Rx_i$, then

$$\sum_{i=1}^{n} y_i^2 + rR\sum_{i=1}^{n} x_i^2 \leq (r+R)\sum_{i=1}^{n} x_i y_i.$$

Equality holds if $rx_i = y_i = Rx_i$ for at least one i.

Theorem 2.7: [6] Let $\chi_1, \chi_2, \ldots, \chi_n$ be the eigenvalues of the graph $(G\oplus S)_r$, containing r self-loops. Then,

$$E(G\oplus S)_r \geq \frac{r + 2m_s - \frac{r^2}{n} + n\left|\chi_n - \frac{r}{n}\right|\left|\chi_1 - \frac{r}{n}\right|}{\left|\chi_n - \frac{r}{n}\right| + \left|\chi_1 - \frac{r}{n}\right|}.$$

Proof: Taking $x_i = 1, y_i = \left|\chi_i - \frac{r}{n}\right|, r = \left|\chi_n - \frac{r}{n}\right|$ and $R = \left|\chi_1 - \frac{r}{n}\right|$ in 2.7, we obtain

$$\sum_{i=1}^{n}\left|\chi_i - \frac{r}{n}\right|^2 + \left|\chi_n - \frac{r}{n}\right|\left|\chi_1 - \frac{r}{n}\right|\sum_{i=1}^{n} 1^2 \leq \left(\left|\chi_1 - \frac{r}{n}\right| + \left|\chi_n - \frac{r}{n}\right|\right)\sum_{i=1}^{n}\left|\chi_i - \frac{r}{n}\right|.$$

Inequality becomes,

$$r + 2m_S - \frac{r^2}{n} + n\left|\chi_n - \frac{r}{n}\right|\left|\chi_1 - \frac{r}{n}\right| \leq \left(\left|\chi_1 - \frac{r}{n}\right| + \left|\chi_n - \frac{r}{n}\right|\right) E(G\oplus S)_r$$

$$E(G\oplus S)_r \geq \frac{r + 2m_s - \frac{r^2}{n} + n\left|\chi_n - \frac{r}{n}\right|\left|\chi_1 - \frac{r}{n}\right|}{\left|\chi_n - \frac{r}{n}\right| + \left|\chi_1 - \frac{r}{n}\right|}.$$

■

Theorem 2.8: Let G be a graph of order n and $\sum_{i<j}\chi_{ij}^2 = m_s$. Then $E(G\oplus S)_r \leq \sqrt{n\left(r + 2m_s - \frac{r^2}{n}\right)}$.

Proof: We have $\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\chi_i - \frac{r}{n}\right| - \left|\chi_j - \frac{r}{n}\right|\right)^2 \geq 0$

$$n\left(\sum_{i=1}^{n}\left|\chi_i - \frac{r}{n}\right|^2 + \sum_{j=1}^{n}\left|\chi_j - \frac{r}{n}\right|^2\right) - 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\chi_i - \frac{r}{n}\right|\left|\chi_j - \frac{r}{n}\right| \geq 0$$

$$n\left[r + 2m_s - \frac{r^2}{n}\right] \geq (E(G\oplus S)_r)^2$$

$$E(G\oplus S)_r \leq \sqrt{n\left(r + 2m_s - \frac{r^2}{n}\right)}.$$

■

III. Energy of Partial Complement of some graphs with self-loops

In this section, we aim to provide the energy of partial complement of various classes of graphs with self-loops. We adopt the eigenvector approach to prove the theorems.

Theorem 3.1: Let $(K_n \oplus S)_r$ be the complete graph with r self-loops then $E(K_n\oplus S)_r = \frac{r}{n} + \sqrt{4nr - 3r^2}$.

Proof:

$A(K_n \oplus S)_r = \begin{bmatrix} 0_{(n-r)\times(n-r)} & I_{(n-r)\times r} \\ I_{r\times(n-r)} & I_{r\times r}\end{bmatrix}_{n\times n}$ is the adjacency matrix of $(K_n\oplus S)_r$.

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order n partitioned conformally with $A(K_n\oplus S)_r$.

Consider

$$det[\chi I - A(K_n\oplus S)_r]\begin{pmatrix} X \\ Y\end{pmatrix} = \begin{bmatrix} \chi I_{(n-r)\times(n-r)}X_{(n-r)\times 1} - J_{(n-r)\times r}Y_{r\times 1} \\ -J_{r\times(n-r)}X_{(n-r)\times 1} + (\chi I - J)_{r\times r}Y_{r\times 1}\end{bmatrix}. \quad (1)$$

Case 1: Let $X = X_j = e_1 - e_j, j = 2, 3, \ldots, (n-r)$ and $Y = 0_{r\times 1}$.

From equation (1), $(\chi I)X_j - J0_{r\times 1} = \chi X_j$.

Then, $\chi = 0$ is an eigenvalue with multiplicity of at least $(n-r-1)$ as there are $(n-r-1)$ independent vectors of the form X_j.

Case 2: Let $X = 0_{(n-r)\times 1}$ and $Y = Y_j = e_1 - e_j, j = 2, 3, \ldots, r$.

From equation (1), $(\chi I - J)Y_j = \chi Y_j$.

So $\chi = 0$ is an eigenvalue with multiplicity of at least $(r-1)$ since there are $r-1$ independent vectors of the form Y_j.

Case 3: Let $X_{(n-r)} = \mathbf{1}_{(n-r)}$ and $Y = \left(\frac{n-r}{\chi - r}\right)\mathbf{1}_r$, where χ is any root of the equation

$$\chi^2 - r\chi - r(n-r) = 0.$$

From equation (1),

$$\chi I 1_{(n-r)} - J_{(n-r)\times r}\left(\frac{r}{\chi - r}\right)1_r$$
$$= \chi 1_{(n-r)} - \left(\frac{r(n-r)}{\chi - r}\right)1_{(n-r)}$$
$$= \left(\chi - \left(\frac{r(n-r)}{\chi - r}\right)\right)1_{(n-r)}.$$

So $\chi = \frac{r+\sqrt{4nr-3r^2}}{2}$ and $\chi = \frac{r-\sqrt{4nr-3r^2}}{2}$ are the eigenvalues with multiplicity of at least one.
Thus the spectrum of $(K_n \oplus S)_r$ is
$\begin{Bmatrix} 0 & \frac{r+\sqrt{4nr-3r^2}}{2} & \frac{r-\sqrt{4nr-3r^2}}{2} \\ n-2 & 1 & 1 \end{Bmatrix}.$
So, $E(K_n \oplus S)_r = \frac{r}{n} + \sqrt{4nr-3r^2}$. ∎

Theorem 3.2: Let $(K_{1,n-1} \oplus S)_r$ be the partial complement of star graph with r self-loops. Then $E(K_{1,n-1} \oplus S)_r = \frac{2n^2-3nr-2r}{n}$.

Proof: Let

$$A(K_{1,n-1}\oplus S)_r = \begin{bmatrix} B_r & 0_{r\times(n-r-1)} & J_{r\times 1} \\ 0_{(n-r-1)\times r} & I_{(n-r-1)} & J_{(n-r-1)\times 1} \\ J_{1\times r} & J_{1\times(n-r-1)} & J_1 \end{bmatrix}_n$$

be the adjacency matrix of $(K_{1,n-1} \oplus S)_r$. Here B is the adjacency matrix of complete sub-graph and J is the matrix with all entries 1.
Step 1: Consider $|\chi I - A(K_{1,n-1} \oplus S)_r|$.
Then by applying row operation $R_i' \longrightarrow R_i - R_{i+1}$, $i = 2, 3, \dots, (r-1), (r+1), \dots, (n-r-2)$ and column operations $C_i' \longrightarrow C_i + C_{i-1} + \dots + C_{r+1}$, $i = (n-r-1), \dots, (r+2)$ and $C_j' \longrightarrow C_j + C_{j-1} + \dots + C_1$, $j = r, r-1, \dots, 2$ on $|\chi I - A(K_{1,n-1} \oplus S)_r|$, we get $(\chi+1)^{r-1}(\chi-1)^{n-r-2}(\chi^2+\chi(r-n))$.
Hence the spectrum of partial complement of star graph with r self-loops is represented by
$\begin{Bmatrix} 0 & 1 & -1 & n-r \\ 1 & n-r-1 & r-1 & 1 \end{Bmatrix}$
and its energy is $E(K_{1,n-1} \oplus S)_r = \frac{2n^2-3nr-2r}{n}$. ∎

Theorem 3.3: Let $(K_{1,n-1} \oplus S)_r$ be the partial complement of star graph with r self-loops attached to the peripheral vertices. Then $E(K_{1,n-1} \oplus S)_r = \sqrt{4n-3}$.

Proof: Let r be the self-loops added to the graph $V - \{v_0\}$. Then
$A(K_{1,n-1}\oplus S)_r = \begin{bmatrix} 0_{1\times 1} & J_{1\times(n-1)} \\ J_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{bmatrix}_{n\times n}$
be the adjacency matrix of $(K_{1,n-1} \oplus S)_r$.
Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order n.
$|\lambda I - A(K_{1,n-1} \oplus S)_r| \begin{pmatrix} X \\ Y \end{pmatrix} =$

$$\begin{bmatrix} \chi I_{1\times 1}X_{1\times 1} - J_{1\times(n-1)}Y_{(n-1)\times 1} \\ -J_{(n-1)\times 1}X_{1\times 1} + (\chi-1)I_{(n-1)\times(n-1)}Y_{(n-1)\times 1} \end{bmatrix}. \quad (2)$$

Case 1: Let $Y = Y_j = e_1 - e_j, j = 2, 3, \dots, (n-1)$ and $X = 0$.
From equation (2), $-J(0) + (\chi-1)IY_j = (\chi-1)Y_j$.
As there are $(n-2)$ independent vectors of the form Y_j, $\chi = 1$ is an eigenvalue with multiplicity of at least $(n-2)$.

Case 2: Let $X = \mathbf{1}$ and $Y = \left(\frac{1}{\chi-1}\right)$, where χ is any root of the equation

$$\chi^2 - \chi - (n-1) = 0.$$

From equation (2),

$$\chi I(1) - J_{1\times(n-1)}\left(\frac{1}{\chi-1}\right)$$
$$= \chi - \left(\frac{n-1}{\chi-1}\right).$$

So, $\chi = \frac{1+\sqrt{4n-3}}{2}$ and $\chi = \frac{1-\sqrt{4n-3}}{2}$ are the eigenvalues with multiplicity of at least one.
Thus the spectrum of $(K_{1,n-1} \oplus S)_r$ is
$\begin{Bmatrix} 0 & \frac{1+\sqrt{4n-3}}{2} & \frac{1-\sqrt{4n-3}}{2} \\ n-2 & 1 & 1 \end{Bmatrix}.$
So, $E(K_{1,(n-1)} \oplus S)_r = \sqrt{4n-3}$. ∎

Theorem 3.4: Let $K_{l,m} \oplus S$ be partial complement of complete bipartite graph with partites V_1 and V_2 of l and m vertices respectively and r be the number of self-loops attached to any one of the partite.Then $E(K_{l,m} \oplus S)_r = n - 2 + \sqrt{l^2 - 4(l-1-lm)}$.

Proof: Let $A(K_{l,m} \oplus S)_r = \begin{bmatrix} B_{l\times l} & J_{l\times m} \\ J_{m\times l} & I_{m\times m} \end{bmatrix}_{n\times n}$ be the adjacency matrix of $(K_{l,m} \oplus S)_r$ with r self-loops.
Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $n = l + m$.
Consider

$$(\chi I - A(K_{l,m} \oplus S)_r)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} ((\chi+1)I - J)X - JY \\ -JX + (\chi-1)IY \end{bmatrix}. \quad (3)$$

Case 1: Let $X = 0_{l\times 1}$ and $Y = e_1 - e_j = Y_j, j = 2, 3, \dots, m$.
Then, from equation 3 $-J(0) + (\chi-1)IY_j = (\chi-1)Y_j$.
So $\chi = 1$ is the eigenvalue with the multiplicity $m-1$ since there are $m-1$ independent vectors in $Y = Y_j$.
Case 2: Let $Y = 0_{m\times 1}$ and $X = X_j = e_1 - e_j, j = 2, 3, \dots, l$.
From equation (3), $[(\chi+1)I]X_j = (\chi+1)X_j$.
So $\chi = -1$ is an eigenvalue with multiplicity of at least $(l-1)$ since there are $l-1$ independent vectors of the form X_j.
Case 3: Let $X = \mathbf{1}_l$ and $Y = \left(\frac{l}{\chi-1}\right)\mathbf{1}_m$, where χ is any root of the equation

$$\chi^2 - l\chi + (l-1-lm) = 0.$$

From equation (3),
$-J\left(\frac{l}{\chi-1}\right)\mathbf{1}_m + [-J + (\chi+1)I]\mathbf{1}_l$
$= (\chi+1)\mathbf{1}_l + -l\mathbf{1}_l - \left(\frac{ml}{\chi-1}\right)\mathbf{1}_l$
$= \frac{(\chi-1)(\chi+1-l) - ml}{\chi-1}\mathbf{1}_l.$
So $\chi = \frac{l}{2} + \frac{\sqrt{(l)^2 - 4(l-1-lm)}}{2}$ and
$\chi = \frac{l}{2} - \frac{\sqrt{(l)^2 - 4(l-1-ml)}}{2}$ are the eigenvalues with multiplicity of at least one. Thus the spectrum of partial complement of complete bipartite graph is

$$\begin{Bmatrix} 1 & -1 & \frac{l+P}{2} & \frac{l-P}{2} \\ m-1 & l-1 & 1 & 1 \end{Bmatrix},$$

where $P = \sqrt{(l)^2 - 4(l-1-lm)}$.

So, $E(K_{l,m} \oplus S)_r = n - 2 + \sqrt{l^2 - 4(l-1-lm)}$. ■

Theorem 3.5: Let $(K_{n\times 2} \oplus S)_r$ be the partial complement of Cocktail party graph with r self-loops to K_n. Then $E(K_{n\times 2} \oplus S)_r = 2n - 2 + \sqrt{n^2 + 4(n-1)^2}$.

Proof: Let $A(K_{n\times 2} \oplus S)_r = \begin{bmatrix} 0_n & B_n \\ B_n & J_n \end{bmatrix}_{2n\times 2n}$ be the adjacency matrix of $(K_{n\times 2} \oplus S)_r$, where B is the adjacency matrix of complete sub-graph.

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $A(K_{n\times 2} \oplus S)_r$.

Consider

$$(\chi I - A)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} \chi IX + (J-I)Y \\ (J-I)X + (\chi I - J)Y \end{bmatrix}. \quad (4)$$

Case 1: Let $X = X_j = e_1 - e_j, j = 2, 3, \ldots, n$ and $Y = \frac{1}{\chi}X_j$, by substituting in equation (4) we get $\chi = -1$ and $\chi = 1$ are the roots with the multiplicity of at least $(n-1)$, as there are $n-1$ eigenvectors of the form X_j.

Case 2: Let $X = 1_n$ and $Y = \frac{-\chi 1_n}{(J-I)}$, where χ is any root of the equation, $\chi^2 - n\chi - (n-1)^2 = 0$.

From equation (4),

$$(J-I)\mathbf{1}_n + (\chi I - J)\frac{-\chi}{J-I}\mathbf{1}_n = \left(\frac{(n-1)^2 - \chi(\chi - n)}{(J-I)}\right)\mathbf{1}_n.$$

Thus $\chi = \frac{n + \sqrt{n^2 + 4(n-1)^2}}{2}$ and $\chi = \frac{n - \sqrt{n^2 + 4(n-1)^2}}{2}$ are the eigenvalues with multiplicity of at least one. Therefore energy of $K_{n\times 2} \oplus S$ with r self-loops is $E(K_{n\times 2} \oplus S)_r = 2n - 2 + \sqrt{n^2 + 4(n-1)^2}$. ■

Theorem 3.6: Let $K_{l,m} \oplus S$ be partial complement of complete bipartite graph with partites V_1 and V_2 of l and m vertices respectively and r self-loop consists of p vertices of V_1 and q vertices of V_2. Then the characteristic polynomial of $[(K_{l,m} \oplus S)_r, \chi] = (\chi+1)^{(p+q-2)}\chi^{(l-p-1)}(\chi-1)^{(m-q-1)}(\chi^4 + (1-q-l)\chi^3 + (p-l-lm+lq+pq-p^2-1)\chi^2 + (l+q+2lm-lp+2lq+mp-2pq-lp^2+l^2p-lq^2-2mp^2+2p^2q+2lmp+lmq-lpq-1)\chi + l-p-lm-lp+lq+mp-pq-lp^2+l^2p-lq^2-2mp^2+pq^2+2p^2q+p^2-p^2q^2+lpq^2+mp^2q+2mlp+lmq-2lpq-mpq-lmpq)$.

Proof: Let $A(K_{l,m} \oplus S)_r =$

$$\begin{pmatrix} (J-I)_{p\times p} & 0_{p\times(l-p)} & 0_{p\times q} & J_{p\times m-q} \\ 0_{(l-p)\times p} & J_{(l-p)\times(l-p)} & J_{(l-p)\times q} & J_{(l-p)\times(m-q)} \\ 0_{q\times p} & J_{q\times(l-p)} & (J-I)_{q\times q} & 0_{q\times(m-q)} \\ J_{(m-q)\times p} & J_{(m-q)\times(l-p)} & 0_{(m-p)\times q} & I_{(m-q)\times(m-q)} \end{pmatrix}_{n\times n}.$$

Consider $|\chi I - A(K_{l,m} \oplus S)_r|$.

On performing row operations $R_i' \longrightarrow R_i - R_{i+1}, i = 2, 3, \ldots, (p-1), (p+1), \ldots, (l-p-1), (l-p+1), \ldots, (q-1), (q+1), \ldots, (m-q-1)$. And column operations $C_i' \longrightarrow C_i + C_{i-1} + \ldots + C_{q+1}, i = (m-q), (m-q-1), \ldots, (q+2)$, $C_j' \longrightarrow C_j + C_{j-1} + \ldots + C_{l-p+1}, j = q, q-1, \ldots, (l-p+2)$, $C_w' \longrightarrow C_w + C_{w-1} + \ldots + C_{p+1}, w = l-p, l-p-1, \ldots, (p+2)$ and $C_z' \longrightarrow C_z + C_{z-1} + \ldots + C_1, z = p, p-1, \ldots, 2$ on $|\chi I - A(K_{1,n-1} \oplus S)_r|$, we get

$|\chi I - A(K_{1,n-1} \oplus S)_r| = (\chi+1)^{(p+q-2)}(\chi)^{(l-p-1)}(\chi-1)^{(m-q-1)}\det(B)$.

We have,

$$\det(B) = \begin{vmatrix} \chi-(p-1) & 0 & 0 & -(l+m-q) \\ 0 & \chi+p-l & -q & -(m-q) \\ 0 & -(l-p) & \chi-(q-1) & 0 \\ -p & -(l-p) & 0 & (\chi-1) \end{vmatrix}$$

which is on expansion leads to the polynomial

$(\chi^4 + (1-q-l)\chi^3 + (p-l-lm+lq+pq-p^2-1)\chi^2 + (l+q+2lm-lp+2lq+mp-2pq-lp^2+l^2p-lq^2-2mp^2+2p^2q+2lmp+lmq-lpq-1)\chi + l-p-lm-lp+lq+mp-pq-lp^2+l^2p-lq^2-2mp^2+pq^2+2p^2q+p^2-p^2q^2+lpq^2+mp^2q+2mlp+lmq-2lpq-mpq-lmpq)$.

Therefore the characteristic polynomial of partial complement of complete bipartite graph is,

$[(K_{l,m} \oplus S)_r, \chi] = (\chi+1)^{(p+q-2)}\chi^{(l-p-1)}(\chi-1)^{(m-q-1)}(\chi^4 + (1-q-l)\chi^3 + (p-l-lm+lq+pq-p^2-1)\chi^2 + (l+q+2lm-lp+2lq+mp-2pq-lp^2+l^2p-lq^2-2mp^2+2p^2q+2lmp+lmq-lpq-1)\chi + l-p-lm-lp+lq+mp-pq-lp^2+l^2p-lq^2-2mp^2+pq^2+2p^2q+p^2-p^2q^2+lpq^2+mp^2q+2mlp+lmq-2lpq-mpq-lmpq)$. ■

Theorem 3.7: Let $(S_n^0 \oplus S)_r$ be the partial complement of a crown graph with r self-loops attached to one of the partites of $V = \{V_1, V_2\}$. Then $E(S_n^0 \oplus S)_r = 2\sqrt{2}(n-1) + \sqrt{n^2 - 4(n-1)(2-n)}$.

Proof: Let $A(G \oplus S)_r = \begin{bmatrix} (J-I)_n & -(J-I)_n \\ (I-J)_n & (I)_n \end{bmatrix}_{2n\times 2n}$ be the adjacency matrix of $(S_n^0 \oplus S)_r$.

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$.

Consider

$$(\chi I - A(G \oplus S)_r)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(\chi+1)I - J]X - (J-I)Y \\ (I-J)X + [(\chi-1)I]Y \end{bmatrix}. \quad (5)$$

Case 1: Let $X = 1_n$ and $Y = \left(\frac{n-1}{\chi-1}\right)1_n$, where χ is any root of the equation

$$\chi^2 - n\chi + (n-1)(2-n) = 0.$$

From equation (5),

$$((\chi+1)I - J)1_n - \frac{(J-I)(n-1)}{(\chi-1)}1_n = \left[(\chi+1-n) - \frac{(n-1)^2}{(\chi-1)}\right]1_n.$$

Thus $\chi = \frac{n + \sqrt{n^2 - 4(n-1)(2-n)}}{2}$ and $\chi = \frac{n - \sqrt{n^2 - 4(n-1)(2-n)}}{2}$ are the eigenvalues with multiplicity of at least one.

Case 2: Let $X = X_j = e_1 - e_j, j = 2, 3, \ldots, n$ and $Y = \frac{-X_j}{\chi-1}$, where χ is root of the equation

$$\chi^2 - 2 = 0.$$

From equation (5),

$$((\chi+1)I - J)X_j + \left(\frac{J-I}{\chi-1}\right)X_j = \left[(\chi+1) - \frac{1}{(\chi-1)}\right]X_j.$$

Hence $\chi = \sqrt{2}$ and $\chi = -\sqrt{2}$ are the eigenvalues each with multiplicity of at least $(n-1)$ as there are $(n-1)$

eigenvectors of the form X_j.
Thus the spectrum of partial complement of crown graph with self-loops is
$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} & \dfrac{n+P}{2} & \dfrac{n-P}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix},$$
where $P = \sqrt{n^2-4(n-1)(2-n)}$ and its energy is $E(S_n^0 \oplus S)_r = 2\sqrt{2}(n-1)+\sqrt{n^2-4(n-1)(2-n)}$. ■

Theorem 3.8: For a partial complement of friendship graph $(F_n \oplus S)_r$ with $r-$self-loops to its peripheral vertices. Then, $E(F_n \oplus S)_r = n + 2\sqrt{n} - 3$.

Proof: Let the adjacency matrix of $(F_n \oplus S)_r$ is
$$A(F_n \oplus S)_r = \begin{bmatrix} 0_{1\times 1} & J_{1\times 2} & J_{1\times 2} & \cdots & J_{1\times 2} \\ J_{2\times 1} & J_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ J_{2\times 1} & 0_{2\times 2} & J_{2\times 2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{2\times 1} & 0_{2\times 2} & 0_{2\times 2} & \cdots & J_{2\times 2} \end{bmatrix}_{\frac{n-1}{2}+1}.$$

The characteristic polynomial of $(F_n \oplus S)_r$ is given by $|\chi I - A(F_n \oplus S)_r| =$
$$\begin{vmatrix} \chi I & -J_{1\times 2} & -J_{1\times 2} & \cdots & -J_{1\times 2} \\ -J_{2\times 1} & (\chi I - J)_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ -J_{2\times 1} & 0_{2\times 2} & (\chi I - J)_{2\times 2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -J_{2\times 1} & 0_{2\times 2} & 0_{2\times 2} & \cdots & (\chi I - J)_{2\times 2} \end{vmatrix}.$$

Step 1: Applying row operation $R_i^{'} \longrightarrow R_i - R_{i+1}$, for $i = 2, 3, \ldots, \left(\frac{n-1}{2} - 1\right)$ and $C_i^{'} \longrightarrow C_i + C_{i-1} + \ldots + C_2$ for $i = \left(\frac{n-1}{2}\right), \left(\frac{n-1}{2} - 1\right), \ldots, 3$ on the above determinant, we get $|\chi I - J|^{\frac{n-1}{2}-1} det(B)$.

Step 2: Further simplification leads to the polynomial $[(\chi-1)^2-1]^{\frac{n-3}{2}}(\chi^3 - 2\chi^2 + \chi(n-1))$.

Hence the spectrum of partial complement of friendship graph with $r-$self-loop $(F_n \oplus S)_r$ is
$$\begin{pmatrix} 0 & 2 & 1-\sqrt{n} & 1+\sqrt{n} \\ \dfrac{n-1}{2} & \frac{n-3}{2} & 1 & 1 \end{pmatrix}.$$

So, $E(F_n \oplus S)_r = n + 2\sqrt{n} - 3$. ■

IV. Concluding remarks

The energy of a graph with self-loops is one of the emerging topic within graph theory. Graph energy has so many application in the field of Chemistry, Physics and Mathematics. In this article, we have derived energy of partial complement of some standard graphs with self-loops and bounds for the same.

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