Generalized Graph Complementation Through Edge Partition

Medini H R, Sabitha D'Souza*, Devadas Nayak C and Pradeep G Bhat

Abstract—**Graph theory is a fundamental area that explores the properties and relationships present within graph structures. This paper explores generalized graph complements by examining edge partitioning, a novel approach that extends the traditional graph complement. In analogy to the vertex partition-based definitions of generalized complements, two distinct types of generalized complements for a graph emerge through edge partitioning: k'-complement and k'(i)-complement of a graph. k'-complement of a graph emerges as a versatile tool with applications across multiple domains, due to its maximum edge configuration linked with a Fibonacci polynomial and its role as a maximal outerplanar graph. The paper systematically examines theorems and fundamental properties governing the structural relationships between the original graph and its generalized complements. The characterizations of cycles and paths are also analysed in this study.**

Index Terms—**Generalized complement, Graph complement, Partition, Cycle, Path**

I. Introduction

In graph theory, a graph is a mathematical representation of a set of objects and the connections between them. The concept of graph complement has been extensively studied in graph theory. The graph complement is obtained by removing the edges of the original graph and adding the edges between previously not connected vertices.

E. Sampathkumar and L. Pushpalatha introduced the notion of the generalized complement of a graph, which they termed as k-complement of a graph [1]. Let P be a partition of the vertex set of G consisting of k partites. To obtain the k-complement of a graph G, remove the edges between the partites which are there in G and add the edges between the partites which are not there in G. The k-complement is denoted as G_k^P.

E. Sampathkumar et al. also found another generalization of the complement of a graph, termed as $k(i)$-complement of a graph [2]. To obtain $k(i)$-complement of a graph G, remove the edges inside each partite which are there in G and add the edges between those vertices which are not there in G. The $k(i)$-complement is denoted as $G_{k(i)}^P$. The authors have extensively worked on $k/k(i)$-self-complementary graphs. Many studies were conducted on the properties, characterizations and applications of the generalized complements of graphs based on the vertex partition [3], [4], [5], [6], [7]. This concept of the generalized complement of a graph is based on the vertex set partition which extends the notion of defining the generalized complement of a graph based on the edge set partition. This new concept has potential applications in diverse fields.

In this paper, a detailed examination is provided for the generalized complements of a graph concerning the partition of edges, exploring its properties, theorems and applications. The findings presented in this paper offer a valuable foundation for further research and exploration of the generalized complement of graphs through edge set partitioning. The Fibonacci sequence continues to be a significant area of research, with ongoing studies exploring its various properties [8]. This study explores the significance of k'-complement of a graph by finding its relation with the maximal outerplanar graph and Fibonacci polynomial by its maximum edge configuration. This connection extends its applicability across diverse fields, encouraging continued investigation and innovation.

II. Generalized complements on edge partition

Consider a simple graph G with vertex set V and edge set E. The k'-complement and k'(i)-complement of a graph are defined as follows:

Let G be a graph without any isolated vertices and $P = \{E_1, E_2, ..., E_k\}$ be a partition of E in which each edge e is represented by its end vertices u and v as $e = uv$.

Definition 1: The k'-complement of a graph G is defined as follows: For all E_i and E_j in P, $i \neq j$, add the edges between E_i and E_j which are not in G. It is denoted as $G_{k'}^P$.

The graph G is k'-self complementary (k'-s.c.) if there exists a partition P of order k such that $G_{k'}^P \cong G$.

The graph G is k'-co-self complementary (k'-co-s.c.) if there exists a partition P of order k such that $G_{k'}^P \cong \overline{G}$.

Definition 2: The k'(i)-complement of a graph G is defined as follows: For each set E_r in P, remove all the edges of G which are inside E_r and add the edges between non-adjacent vertices inside E_r. It is denoted as $G_{k'(i)}^P$.

The graph G is k'(i)-self complementary (k'(i)-s.c.) if there exists a partition P of order k such that $G_{k'(i)}^P \cong G$.

The graph G is k'(i)-co-self complementary (k'(i)-co-s.c.) if there exists a partition P of order k such that $G_{k'(i)}^P \cong \overline{G}$.

Note 1: The common vertex belonging to different edge sets in P are considered inside to their respective edge sets.

u is adjacent to v is denoted as $u \sim v$.

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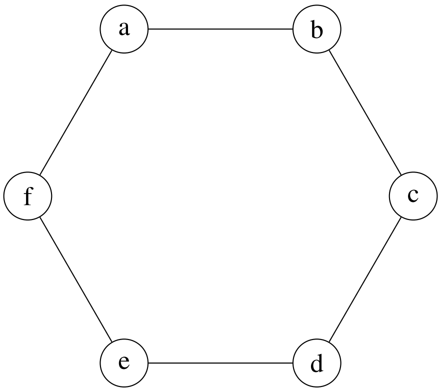

Fig. 1. Cycle C_6

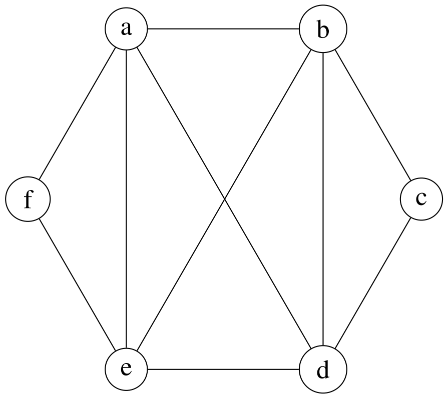

Fig. 2. 2'-complement $G^P_{2'}$

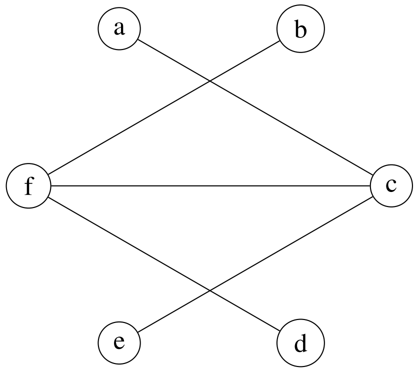

Fig. 3. 2'(i)-complement $G^P_{2'(i)}$

Example 1: Consider the cycle C_6 (Fig. 1). Let P = $\{E_1, E_2\}$, where E_1 = {ab, bc, af} and E_2 = {cd, de, ef}. The vertices c and f are present in both edge sets. So, they are considered as inside to both E_1 and E_2.

In $G^P_{2'}$, the edges inside E_1 and E_2 remain the same. The edges are joined from the vertices of E_1 to the vertices of E_2 which are not in E_1 as shown in Fig. 2.

In $G^P_{2'(i)}$, the edges inside E_1 and E_2 are removed. The edges are then joined between non-adjacent vertices inside E_1 and E_2 as shown in Fig. 3.

A. Properties of k'-complement and $k'(i)$-complement of graphs

Let G be a simple graph with vertex set V of order n and edge set E of size m, then the following properties hold:

i) $G^P_{1'} \cong G$ and $G^P_{1'(i)} \cong \overline{G}$,

ii) $G^P_{m'} \cong K_n$ and $G^P_{m'(i)} \cong N_n$, where P is the partition of E into singleton sets, K_n is the complete graph of order n and N_n is the null graph of order n.

The k'-complement and $k'(i)$-complement are related as follows:

Proposition 1: For any graph G,

$$\text{i) } \overline{G^P_{k'}} \cong G^P_{k'(i)}$$

$$\text{ii) } \overline{G^P_{k'(i)}} \cong G^P_{k'}$$

Proof: Let u and v be two vertices of G.

i) Then $u \sim v$ in $\overline{G^P_{k'}}$
$\iff$ u and v belong to the same edge set in P and non-adjacent in G
$\iff$ $u \sim v$ in $G^P_{k'(i)}$.
$\implies$ $\overline{G^P_{k'}} \cong G^P_{k'(i)}$.

ii) Then $u \sim v$ in $\overline{G^P_{k'(i)}}$
$\iff$ u and v are in the different edge sets and non-adjacent in G or they belong to the same edge set and adjacent in G
$\iff$ $u \sim v$ in $G^P_{k'}$.
$\implies$ $\overline{G^P_{k'(i)}} \cong G^P_{k'}$. ■

From Proposition 1, we have

Corollary 0.1: $G^P_{k'} \cong G \iff G^P_{k'(i)} \cong \overline{G}$.
In other words, G is k'-s.c. if and only if G is $k'(i)$-co.s.c.

Corollary 0.2: $G^P_{k'(i)} \cong G \iff G^P_{k'} \cong \overline{G}$ only when $G \cong \overline{G}$. In other words, G is $k'(i)$-s.c. if and only if G is k'-co-s.c. and self-complementary (G is s.c.).

Proof: From the definition, the edges of $G^P_{k'}$ correspond to the edges of G. So, if $G^P_{k'(i)} \cong G$ and $G^P_{k'} \cong \overline{G}$, then $G^P_{k'} \cong G$. So, $G \cong \overline{G}$. ■

Remark 1: i). Due to the different edge sets in G and $\overline{G}$, the partitions in G and $\overline{G}$ are different. So, in general, $\overline{G^P_{k'}} \ncong \overline{G}^P_{k'}$. Similarly, $\overline{G^P_{k'(i)}} \ncong \overline{G}^P_{k'(i)}$.

ii). Let G and $\overline{G}$ be graphs without any isolated vertices. Let P and Q be the partitions of the edge sets in G and $\overline{G}$ respectively.
When k = 1, $G^P_{1'} \cong G$ and $\overline{G}^Q_{1'} \cong \overline{G}$.
But $G^P_{1'} \cong G \iff \overline{G^P_{1'}} \cong \overline{G}$.
$\implies \overline{G^P_{1'}} \cong \overline{G}^Q_{1'}$

Similarly, $G^P_{1'(i)} \cong \overline{G}$ and $\overline{G}^Q_{1'(i)} \cong G$.
But $G^P_{1'(i)} \cong \overline{G} \iff \overline{G^P_{1'(i)}} \cong G$.
$\implies \overline{G^P_{1'(i)}} \cong \overline{G}^Q_{1'(i)}$.

Proposition 2: Let G and $\overline{G}$ be graphs without any isolated vertices. Let P and Q be the partitions of the edge sets in G and $\overline{G}$ respectively. If G is not k'-s.c., then $\overline{G^P_{k'}} \ncong \overline{G}^Q_{k'}$ and $\overline{G^P_{k'(i)}} \ncong \overline{G}^Q_{k'(i)}$, for $k > 1$.

Proof: Let $k > 1$.
If G is not k'-s.c., then $G^P_{k'}$ contains all the edges of G and at least one edge of $\overline{G}$. Hence, $\overline{G^P_{k'}}$ contains at least one edge less than $\overline{G}$. But $\overline{G}^Q_{k'}$ has all the edges of $\overline{G}$. So, $\overline{G^P_{k'}} \ncong \overline{G}^Q_{k'}$. Similarly, $\overline{G^P_{k'(i)}} \ncong \overline{G}^Q_{k'(i)}$. ■

Proposition 3: Let G and $\overline{G}$ be graphs without any isolated vertices.Then $\overline{G^P_{k'}} \cong \overline{G}^Q_{k'}$ and $\overline{G^P_{k'(i)}} \cong \overline{G}^Q_{k'(i)}$ only when G and $\overline{G}$ are k'-s.c. with respect to P and Q respectively.

Proof: Case 1: Suppose G is not k'-s.c., then the proof follows from Proposition 2.

Case 2: Suppose $\overline{G}$ is not k'-s.c. Let G be k'-s.c. Then, $G^P_{k'}$ only contains the edges of G. Thus, $\overline{G^P_{k'}}$ only contains the edges of $\overline{G}$.

On the other hand, since $\overline{G}$ is not k'-s.c. $\overline{G}^Q_{k'}$ contains all the edges of $\overline{G}$ and at least one edge from G. But $\overline{G^P_{k'}}$ only contains the edges of $\overline{G}$. In this case, $\overline{G^P_{k'}} \ncong \overline{G}^Q_{k'}$ which is a contradiction.

That is, if G is k'-s.c. with respect to P, then $G^P_{k'} \cong G \iff \overline{G^P_{k'}} \cong \overline{G}$.

If $\overline{G}$ is k'-s.c. with respect to Q, then $\overline{G}^Q_{k'} \cong \overline{G}$. Thus, $\overline{G^P_{k'}} \cong \overline{G}^Q_{k'}$.

Similarly, $\overline{G^P_{k'(i)}} \cong \overline{G}^Q_{k'(i)}$. ■

Corollary 0.3: Let G and $\overline{G}$ be graphs without any isolated vertices. Then $G^P_{k'(i)} \cong \overline{G}^Q_{k'}$ and $G^P_{k'} \cong \overline{G}^Q_{k'(i)}$ only when G and $\overline{G}$ are k'-s.c. with respect to P and Q respectively.

Proposition 4: Every $G^P_{k'(i)}$ is a subgraph of $\overline{G}$.

Proof: We know that $G^P_{k'}$ includes all the edges of G. From Proposition 1, it follows that $\overline{G^P_{k'}} \cong G^P_{k'(i)}$. As a result, the edges in $G^P_{k'(i)}$ correspond to the edges of $\overline{G}$. Hence, $G^P_{k'(i)}$ is a subgraph of $\overline{G}$. ■

Observations:

1. For any graph G free from isolated vertices, $G^P_{k'}$ can never be a null graph.

2. If G is a complete graph, then $G^P_{k'}$ is also a complete graph and G is k'-s.c. with respect to any P.

3. If G is a complete graph, then $G^P_{k'(i)}$ is a null graph and G is $k'(i)$-co-s.c. with respect to any P.

4. Let G and $\overline{G}$ be graphs without any isolated vertices. Then $G^P_{m'} \cong \overline{G}^Q_{m'} \cong K_n$ and $G^P_{m'(i)} \cong \overline{G}^Q_{m'(i)} \cong N_n$, where m is the number of edges taken as singleton sets in P.

B. Characterization of k'-self complementary graph

Theorem 1: If union of edges in any $E_i \in P$ is a spanning subgraph of G, then G is k'-s.c.

In other words, if any $E_i \in P$ is an edge covering set, then $G^P_{k'} \cong G$.

Proof: Let P = $\{E_1, E_2,...,E_k\}$ be a partition of E. If the union of edges in any $E_i \in P$ is a spanning subgraph of G, then all the vertices lie inside E_i and no extra edge can be added to G. Since k'-complement preserves the original graph's connectivity, G will be k'-s.c.

If any $E_i \in P$ is an edge covering set, then E_i covers all the vertices of G and the vertices lie inside E_i. Hence, no more edges can be added to G. Thus, G is k'-s.c. ■

Example 2: Consider the Petersen graph G.

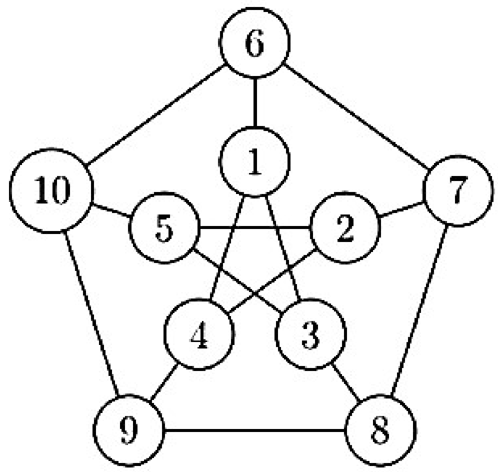

Fig. 4. Petersen graph G

Let P = $\{E_1, E_2, E_3, E_4\}$
E_1 = {(1,3), (5,2), (2,4), (6,7), (9,8), (9,10)}
E_2 = {(1,4), (3,5), (1,6)}
E_3 = {(7,8), (6,10), (2,7)}
E_4= {(3,8), (4,9), (5,10)}
Since E_1 covers all the vertices of G, it is $4'$-s.c.

Here, we can find the smallest edge covering set to obtain k'-s.c. graph. Let P = $\{E_1, E_2\}$, where E_1 = {(1,6), (2,7), (3,8), (4,9), (5,10)}. The set E_1 has five edges covering all the vertices of G. Hence, it is the smallest edge covering set. Thus, the Petersen graph is a $2'$-self complement.

Remark 2: G is $2'$-s.c. if and only if either E_1 or E_2 in P is an edge covering set.

The converse of Theorem 1 holds in cycle with k = 2, path and trees. But the converse need not be true in general.

Example 3: Let G be $3'$-s.c. Consider P = $\{E_1, E_2, E_3\}$, where E_i, i = 1, 2, 3 is not an edge covering set. That is E_i fails to cover at least one vertex of G.

Consider the cycle C_5 as shown in Fig. 5. Let P = $\{E_1, E_2, E_3\}$, where E_1 = {ab, ed} and $c \notin V_{E_1}$, E_2 = {ae, bc} and $d \notin V_{E_2}$, E_3 = {cd}. With respect to the partition P, it is clear that C_5 is $3'$-s.c. even when any $E_i \in P$ is not an edge covering set.

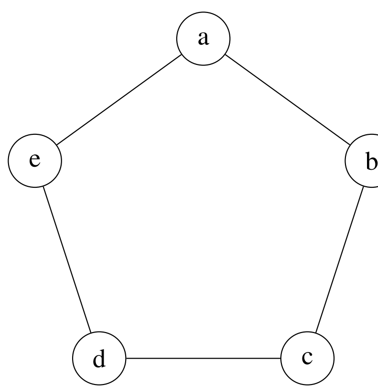

Fig. 5. Cycle C_5

C. Characterization of cycle C_n

Theorem 2: Let $G = C_n$ be k'-s.c. If any $E_i \in P$ is not an edge covering set, then k = 3.

Proof: Let C_n be k'-s.c. and each $E_i \in P$ be a non-edge covering set. Then, each E_i does not cover at least one vertex of G. Since each vertex of a cycle is adjacent to two vertices, every vertex belongs to at most two edge sets.

Case 1: A vertex of C_n is taken twice in only one edge set. Suppose v_i is taken twice in E_i of P. Let $v_j \notin V_{E_i}$. Since v_j is lying outside E_i and v_i is present only in E_i, an edge will be added between v_i and v_j. Then the resultant graph is not k'-s.c. Hence, the adjacent edges cannot lie in only one edge set.

Case 2: Each vertex of C_n is taken in two edge sets.
Let v_1 belong to E_1 and E_2. Suppose v_2 is lying outside of both E_1 and E_2. Then in k'-complement, an edge is added between v_1 and v_2. So, no vertex should lie outside of both E_1 and E_2. Thus, E_1 and E_2 should cover all the vertices in G. This means both E_1 and E_2 together contain each vertex at least once and those vertices can be considered again at most once. Since, E_1 and E_2 together cover all vertices of G, the vertices lying outside E_1 will be there in E_2 and vice versa. So, those vertices can be taken again once in another edge set.

Suppose the vertices lying outside E_1 and E_2 are taken in different edge sets, then an edge will be added between them. In order to avoid the addition of edges to the graph, the vertices lying outside E_1 and E_2 should be present together in one edge set E_3. Hence, the maximum possible number of partitions to obtain k'-s.c. is 3 when P does not contain an edge covering set. ■

Corollary 2.1: Let C_n be 3'-s.c. If any $E_i \in P$ is not an edge covering set, then the union of any two E_i is an edge covering set.

Example 4: Let $P = \{E_1, E_2, E_3\}$ in C_7, where E_1 = {ab, cd}, E_2 = {ag, ef} and E_3 = {bc, de, fg}. So, C_7 is 3'-s.c.
$E_1 \cup E_2$ = {ab, cd, ag, ef} is an edge covering set
$E_2 \cup E_3$ ={ag, ef, bc, de, fg} is an edge covering set.
$E_1 \cup E_3$ = {bc, de, fg, ab, cd} is also an edge covering set.

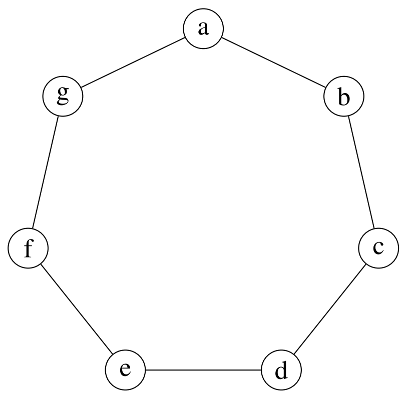

Fig. 6. Cycle C_7

Corollary 2.2: Let each $E_i \in P$ be a non-edge covering set. For $k > 3$, C_n is k'-s.c. only when C_n contains chords.

Proof: If C_n contains a chord, then there is an increased possibility of selecting vertices for each edge set in a partition. This happens because the adjacent vertices of the chord can be taken more than twice. ■

Example 5: In the graph shown below, let $P = \{E_1, E_2, E_3, E_4\}$, where E_1 = {ad}, E_2 = {ab, ac, ae}, E_3= {de, ef}, E_4 = {af, bc, cd}. Here, each vertex is related to every other vertex of the graph within the edge sets, ensuring all vertices are contained within and no further edge can be added to the graph in k'-complement. So, the graph is 4'-s.c. with respect to P.

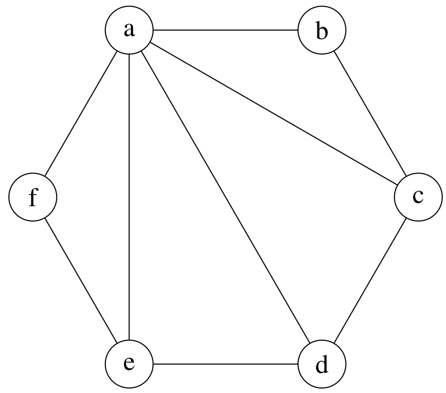

Fig. 7. C_6 with 3 chords

Theorem 3: For $G = C_n$, the bounds for $m(G^P_{2'})$, where $m(G^P_{2'})$ is the size of $G^P_{2'}$ are given by,

$$n \leq m(G^P_{2'}) \leq \frac{n^2+3}{4}, \quad \textit{if } n \textit{ is odd,}$$

$$n \leq m(G^P_{2'}) \leq \frac{n^2}{4} + 1, \quad \textit{if } n \textit{ is even.}$$

Proof: The edges of C_n remain unchanged in k'-complement. Hence, 2'-complement will have at least n edges.

For an odd cycle, the maximum number of edges possible in 2'-complement is $(\frac{n-1}{2})(\frac{n-1}{2} - 1)$ = $(\frac{n^2-4n+3}{4})$.
Thus, the number of edges in an odd cycle is at most n + $(\frac{n^2-4n+3}{4})$ = $\frac{n^2+3}{4}$.

For an even cycle, the maximum number of edges possible in 2'-complement is $(\frac{n}{2} - 1)^2$.
Thus, the number of edges in an even cycle is at most n + $(\frac{n}{2} - 1)^2$ = $\frac{n^2}{4} + 1$.

■

Corollary 3.1: For $G = C_n$, the bounds for $m(G^P_{2'(i)})$ are given by,

$$\frac{(n-3)(n+1)}{4} \leq m(G^P_{2'(i)}) \leq \frac{n(n-3)}{2}, \quad \textit{if } n \textit{ is odd,}$$

$$\frac{n^2-2n-4}{4} \leq m(G^P_{2'(i)}) \leq \frac{n(n-3)}{2}, \quad \textit{if } n \textit{ is even.}$$

Proof: In C_n, the number of edges in 2'(i)-complement is at most $\frac{n(n-1)}{2} - n = \frac{n^2-3n}{2} = \frac{n(n-3)}{2}$.

Case 1: Let n be odd in C_n.
From Theorem 3, 2'-complement graph can have at

most $\frac{n^2+3}{4}$ edges. Hence, the number of edges in 2'(i)-complement is at least $\frac{n(n-1)}{2} - (\frac{n^2+3}{4}) = \frac{n^2-2n-3}{4} = \frac{(n-3)(n+1)}{4}$.

Case 2: Let n be even in C_n.
From Theorem 3, 2'-complement graph can have at most $\frac{n^2}{4} + 1$ edges. Hence, the number of edges in 2'(i)-complement is at least $\frac{n(n-1)}{2} - (\frac{n^2}{4} + 1) = \frac{n^2-2n-4}{4}$. ■

D. Construction of k'-s.c. of cycle C_n

The k'-s.c. graph can be obtained by finding the edge covering set of the graph (Theorem 1). The size of the smallest edge covering set (edge covering number $\beta'(G)$) gives the minimum possible number of edges that can be taken in an edge set to obtain k'-s.c. graph.

Theorem 4: Let $G = C_n$ be k'-s.c. If E_i is an edge covering set in P, then E_i will have at least $\lceil \frac{n}{2} \rceil$ edges.

Proof: Let G be k'-s.c. and E_i be the smallest edge covering set in P.
When n is even, $\beta'(C_n) = \frac{n}{2}$ and when n is odd, $\beta'(C_n) = \frac{n+1}{2}$.
Therefore, the size of the smallest edge covering set E_i is $\lceil \frac{n}{2} \rceil$. ■

Corollary 4.1: Let $G = C_n$ be 2'-s.c. and suppose P contains the smallest edge covering set. Then the bounds for cardinality of the edge sets are given by,

$$when\ n\ is\ odd,\ \frac{n+1}{2} \le |E_1| \le n-1,$$
$$1 \le |E_2| \le \frac{n-1}{2},$$
$$when\ n\ is\ even,\ \frac{n}{2} \le |E_1| \le n-1,$$
$$1 \le |E_2| \le \frac{n}{2}.$$

Proposition 5: If $P = \{E_1, E_2\}$ is a partition of C_n such that $|E_1| = 1$ and $|E_2| = m - 1$, then C_n is 2'-s.c.

Proof: Given $|E_1| = 1$. Let $E_1 = \{e\}$, where $e = uv$. The remaining $m - 1$ edges are taken in E_2 in which the remaining path starts with u and ends with v. Hence, E_2 is an edge covering set. Thus, G is 2'-s.c. ■

Observations:

1. In C_n, if $|E_1| = 1$ and $|E_2| = m - 1$, then $G^P_{2'(i)} \cong \overline{G}$.

2. In C_n, if $|E_1| = 2$ with two non-adjacent edges and $|E_2| = m - 2$, then G is 2'-s.c.

3. In C_n, $G^P_{2'}$ obtained from the edge sets $|E_1| = 2$ with two adjacent edges and $|E_2| = m - 2$, is similar to $G^P_{3'}$ obtained from the edge sets $|E_1| = 1$, $|E_2| = 1$ (the edges in E_1 and E_2 are adjacent) and $|E_3| = m - 2$.

E. Characterization of path P_n

Theorem 5: P_n is k'-s.c. if and only if one of the edge sets in P is an edge covering set.

Proof: Let P_n be k'-s.c. and $v \in V_{E_i}$ be a pendant vertex. Since v is adjacent to only one vertex, all the vertices should lie in E_i. Otherwise, the vertices which are not in E_i will be outside to v. As a result, the edges are added between them resulting in a non-k'-s.c. graph. Hence, E_i should cover all the vertices which is an edge covering set.

Conversely, if one of the edge sets in P is an edge covering set, then P_n is k'-s.c. (Theorem 1). ■

Corollary 5.1: Tree is k'-s.c. if and only if one of the edge sets in P is an edge covering set.

Corollary 5.2: A forest is k'-s.c. if and only if one of the edge sets in P is an edge covering set.

Corollary 5.3: A star $K_{1,n}$ is k'-s.c. if and only if all of its edges lie in only one edge set in P ($k = 1$).

Theorem 6: For $G = P_n$, the bounds for $m(G^P_{2'})$, $n > 2$ are given by,

$$n-1 \le m(G^P_{2'}) \le n-1+\frac{n(n-2)}{4},\quad if\ n\ is\ even,$$
$$n-1 \le m(G^P_{2'}) \le n-1+\left(\frac{n-1}{2}\right)^2,\quad if\ n\ is\ odd.$$

Proof: Let $P = \{E_1, E_2\}$. The edges of P_n remain unchanged in 2'-complement and the number of edges in P_n is $n-1$. Hence, 2'-complement graph will have at least $n-1$ edges.

Case 1: Let n be even. The maximum number of edges is obtained when the edges are divided into two partitions where one edge set contains $P_{\frac{n}{2}+1}$ path and another contains the remaining $P_{\frac{n}{2}}$ path. Then only one vertex say u is present in both E_1 and E_2. The cardinality of one edge set is $\frac{n}{2}$ and another edge set is $\frac{n}{2} - 1$. In 2'-complement, the edges are joined from each vertex of E_1 to each vertex of E_2 except u.
Thus, 2'-complement can have at most $(n-1) + \frac{n(n-2)}{4}$ edges when n is even in P_n.

Case 2: Let n be odd. The maximum number of edges is obtained when the edges are evenly divided into two partitions where each edge set contains $P_{\frac{n+1}{2}}$ path. Then only one vertex say u is present in both E_1 and E_2. The cardinality of each edge set is $\frac{n-1}{2}$. In 2'-complement, the edges are joined from each vertex of E_1 to each vertex of E_2 except u.
Thus, 2'-complement can have at most $(n-1) + \left(\frac{n-1}{2}\right)^2$ edges when n is odd in P_n. ■

Let $G^P_{2'}$ have the maximum number of edges and $n > 2$ in P_n.

Corollary 6.1: $m(G^P_{2'})$ in P_n = $m(G^P_{2'})$ in $P_{n-1} + \lceil \frac{n}{2} \rceil$.

Corollary 6.2: $m(G^P_{2'})$ in P_n = $m(G^P_{2'})$ in $P_{n-2} + n$.

From Corollaries 6.1 and 6.2, we have

Corollary 6.3: $m(G^P_{2'})$ in $P_{n-1} - m(G^P_{2'})$ in $P_{n-2} = \lfloor \frac{n}{2} \rfloor$.

Note 2: i) Let n be even in P_n and $n > 2$. The maximum number of edges possible in $G^P_{2'}$ is $n-1+\frac{n(n-2)}{4}$ (Theorem 6). A few terms of this sequence are listed below.

TABLE I
MAXIMUM POSSIBLE NUMBER OF EDGES IN $G^P_{2'}$

n	$m(G^P_{2'})$	n	$m(G^P_{2'})$	n	$m(G^P_{2'})$
4	5	20	109	36	341
6	11	22	131	38	379
8	19	24	155	40	419
10	29	26	181	42	461
12	41	28	209	44	505
14	55	30	239	46	551
16	71	32	271	48	599
18	89	34	305	50	649

ii) Upon analysis of the sequence when n is even in P_n, it becomes evident that a considerable number of terms are prime. However, as n increases, the occurrence of prime terms becomes less frequent.

iii) It is observed that when n is even and $n > 2$ in P_n, the sequence of terms in $n-1+\frac{n(n-2)}{4}$ follows one of the Fibonacci polynomials $n^2 - n - 1$ [9].

Corollary 6.4: When n is even in P_n, $G^P_{2'}$ can have at most $(n-1)+\frac{n(n-2)}{4}$ edges. When $n = 2(n-1)$, the pattern follows Fibonacci polynomial $n^2 - n - 1$.

Proof: Let n be even in P_n and $n > 2$. The maximum number of edges possible in $G^P_{2'}$ is $n-1+\frac{n(n-2)}{4}$ (Theorem 6).

Put $n = 2(n-1)$.

Then,

$$\begin{aligned} n-1+\frac{n(n-2)}{4} &= 2n-2-1+\frac{(2n-2)(2n-4)}{4} \\ &= 2n-3+\frac{4n^2-8n-4n+8}{4} \\ &= 2n-3+n^2-2n-n+2 \\ &= n^2-n-1 \end{aligned}$$

which is a Fibonacci polynomial. ■

Note 3: The above sequence is also similar to the sequences $n+(n+1)^2$ and $|1-n-n^2|$.

Corollary 6.5: The sequence of the maximum number of edges of P_n in $G^P_{2'}$ is similar to the sequence of the maximum number of edges of C_n in $G^P_{2'(i)}$.

Corollary 6.6: For $G = P_n$, n > 2, the bounds for $m(G^P_{2'(i)})$ are given by,

$$\frac{(n-2)^2}{4} \le m(G^P_{2'(i)}) \le \frac{(n-2)(n-1)}{2}, \quad \textit{if } n \textit{ is even},$$

$$\frac{(n-3)(n-1)}{4} \le m(G^P_{2'(i)}) \le \frac{(n-2)(n-1)}{2}, \textit{ if } n \textit{ is odd}.$$

Proof: In P_n, the number of edges in 2'(i)-complement graph is at most $\frac{n(n-1)}{2} - (n-1) = \frac{n^2-3n+2}{2} = \frac{(n-2)(n-1)}{2}$.

Case 1: Let n be even.
From Theorem 6, 2'-complement graph can have at most $(n-1)+\frac{n(n-2)}{4}$ edges. Hence, the number of edges in 2'(i)-complement is at least $\frac{n(n-1)}{2} - (\frac{n^2+2n-4}{4}) = \frac{n^2-4n+4}{4} = \frac{(n-2)^2}{4}$.

Case 2: Let n be odd.
From Theorem 6, 2'-complement graph can have at most $(n-1)+\left(\frac{n-1}{2}\right)^2$ edges. Hence, the number of edges in 2'(i)-complement is at least $\frac{n(n-1)}{2} - (\frac{n^2+2n-3}{4}) = \frac{n^2-4n+3}{4} = \frac{(n-3)(n-1)}{4}$. ■

Note 4: i) When n is odd in P_n, the sequence $\frac{(n-3)(n-1)}{4}$ follows the sequence generated by adding consecutive even numbers starting from 2.

ii) When n is even in P_n, the sequence $\frac{(n-2)^2}{4}$ follows the sequence generated by squaring the natural numbers.

Theorem 7: In P_n $(n > 2)$, if $E_1 = \{e\}$, where e is a pendant edge, then $m(G^P_{2'}) = 2n - 3$.

Proof: Let $P = \{E_1, E_2\}$ and $E_1 = \{e\}$, where $e = uv$ be a pendant edge and v be a pendant vertex. Then, $u \in V_{E_1}$ and $u \in V_{E_2}$. The edges are added from v to the remaining vertices in P_n except u and the number of edges in P_n is $n-1$ which remains in k'-complement. Thus, the number of edges in $G^P_{2'}$ is $n-1+n-2 = 2n-3$. ■

Corollary 7.1: In P_n, if $E_1 = \{e\}$, where e is a pendant edge, then $G^P_{2'}$ is a maximal outerplanar graph.

Proof: Let $E_1 = \{e\}$, where e is a pendant edge in P_n. From Theorem 7, the edges are added from a pendant vertex to the remaining vertices in P_n, except to its adjacent vertex. Then in $G^P_{2'}$, all its vertices lie on the same face and each face is bounded by a triangle. Hence, $G^P_{2'}$ is a maximal outerplanar graph. ■

Note 5: Every maximal outerplanar graph G with n points has $2n - 3$ edges [10].

Note 6: $G^P_{2'}$ is plane triangulation since each face is triangular.

Example 6: Let $P = \{E_1, E_2\}$, where E_1 contains a pendant edge and E_2 contains the remaining edges of P_5 as shown in Fig. 8. In 2'-complement, the edges are joined from the pendant vertex in E_1 to all the remaining vertices in E_2 as shown in Fig. 9. The number of edges in $G^P_{2'}$ of P_5 is $2n - 3 = 7$. The graph in Fig. 9 can also be drawn as the graph in Fig. 10 since both are isomorphic. The graph in Fig. 10 is a maximal outerplanar graph and it shows that each face is bounded by a triangle (plane triangulation).

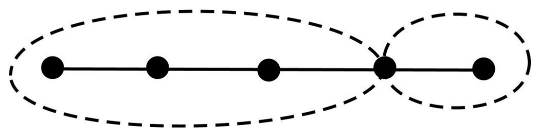

Fig. 8. P_5

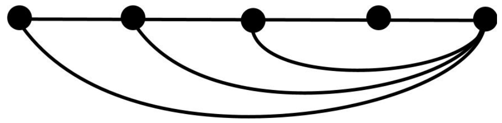

Fig. 9. $G^P_{2'}$

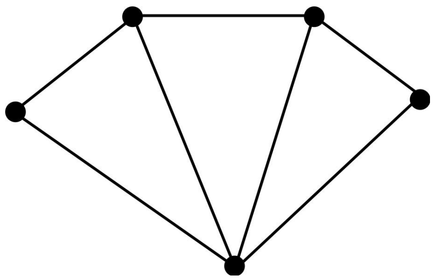

Fig. 10. Maximal outerplanar graph

Corollary 7.2: In P_n, if $E_1 = \{e\}$, where e is a pendant edge, then $m(G^P_{2'(i)}) = \frac{(n-3)(n-2)}{2}$.

Proof: We know that the number of edges in a complete graph is $\frac{n(n-1)}{2}$. When $E_1 = \{e\}$, where e is a pendant edge in P_n, the number of edges in $G^P_{2'}$ is $2n-3$. Hence, the number of edges in $G^P_{2'(i)}$ is $\frac{n(n-1)}{2} - (2n-3) = \frac{n^2-5n+6}{2} = \frac{(n-3)(n-2)}{2}$. ■

Example 7: Consider the partitions of P_5 as shown in Fig. 8. In $2'(i)$-complement of P_5, the edges of P_5 are removed and the edges are added between non-adjacent vertices inside E_2 as shown in Fig. 11. The number of edges in $G^P_{2'(i)}$ of P_5 is $\frac{(n-3)(n-2)}{2} = 3$.

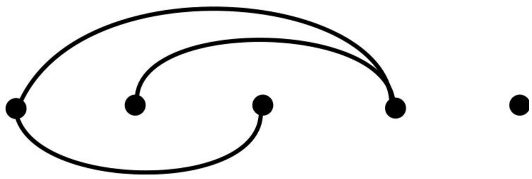

Fig. 11. $G^P_{2'(i)}$

Corollary 7.3: Let $n > 3$ in P_n. If $E_1 = \{e\}$, where e is a pendant edge, then the sequence $\frac{(n-3)(n-2)}{2}$ of edges in $G^P_{2'(i)}$ follows the sequence $\frac{n(n+1)}{2}$ which is the sum of first n natural numbers.

Proof: Let $m = \frac{(n-3)(n-2)}{2}$ in $G^P_{2'(i)}$. Since P_1 contains a single point and P_2 contains a single edge, it is impossible to consider the partitions in 2'-complement. Since P_3 contains two edges, $G^P_{2'(i)}$ is a null graph.

Put $n = n+3$ in $\frac{(n-3)(n-2)}{2}$ for $n > 3$ in P_n. Then it follows the sequence $\frac{n(n+1)}{2}$. ■

F. Construction of k'-s.c. of path P_n

Theorem 8: Let $G = P_n$ be k'-s.c. If E_i is an edge covering set in P, then E_i will have at least $\lceil \frac{n}{2} \rceil$ edges.

Proof: Let G be k'-s.c. and E_i be the smallest edge covering set in P.
When n is even, $\beta'(P_n) = \frac{n}{2}$ and when n is odd, $\beta'(P_n) = \frac{n+1}{2}$.
Therefore, the size of the smallest edge covering set E_i is $\lceil \frac{n}{2} \rceil$. ■

Corollary 8.1: Let $G = P_n$ be 2'-s.c. and suppose P contains the smallest edge covering set. Then the bounds for cardinality of the edge sets are given by,

$$when\ n\ is\ odd,\ \frac{n+1}{2} \leq |E_1| \leq n-2$$
$$2 \leq |E_2| \leq \frac{n-3}{2},$$
$$when\ n\ is\ even,\ \frac{n}{2} \leq |E_1| \leq n-2$$
$$2 \leq |E_2| \leq \frac{n-2}{2}.$$

Proof: Case 1: Let n be odd.
From Theorem 8, E_1 in P has at least $\frac{n+1}{2}$ edges. Then the number of edges in E_2 is at most $m - \frac{n+1}{2}$. In P_n, $m = n-1$.
Thus, $m - \frac{n+1}{2} = n - 1 - (\frac{n+1}{2}) = \frac{n-3}{2}$.

Case 2: Let n be even.
From Theorem 8, E_1 in P has at least $\frac{n}{2}$ edges. Then the number of edges in E_2 is at most $m - \frac{n}{2}$. In P_n, $m = n-1$.
Thus, $m - \frac{n}{2} = n - 1 - \frac{n}{2} = \frac{n-2}{2}$. ■

Proposition 6: In P_n, if $|E_1| = 1$ and $|E_2| = m-1$, then G is 2'-s.c. only when E_1 contains a middle edge.

Proof: Let G be 2'-s.c. and E_1 = {e}, where $e = uv$. The remaining $m-1$ edges are taken in E_2. Suppose e is a pendant edge and v is a pendant vertex lying outside E_2. So, the edges are joined from v to its non-adjacent vertices in G. Hence, G is not 2'-s.c. which is a contradiction. ■

Proposition 7: Every $G^P_{k'}$ of P_n contains a cycle except k'-s.c graphs.

Proof: If G is not k'-s.c., then k'-complement has at least one edge more than G. Initially, P_n has exactly one path between each pair of vertices. Thus, when an edge is joined between non-adjacent vertices in P_n, creates a cycle. ■

G. k'-s.c. of a unicyclic graph

Theorem 9: Let G be a unicyclic graph having exactly one cycle and the remaining edges forming a tree structure extending from the cycle. Then G is k'-s.c. if and only if P contains an edge covering set of G.

Proof: Given that G contains a pendant vertex, any edge set E_i in P containing the pendant edge must cover all vertices to obtain k'-s.c. of G. This makes E_i an edge covering set.

The converse follows directly from Theorem 1. ■

Theorem 10: The minimum possible number of edges in an edge set to obtain k'-s.c. of a unicyclic graph G is $\lceil \frac{n}{2} \rceil$.

Proof: Let G be k'-s.c. and E_i in P be the smallest edge covering set. Since a unicyclic graph G with n vertices has exactly n edges, the edge covering number of G is $\beta'(G) = \lceil \frac{n}{2} \rceil$. ■

Example 8: Consider the unicyclic graph G as shown in Fig. 12. Let $E_i = \{ae, bf, ch, dg\}$ in P. Then E_i is the smallest edge covering set of G and the cardinality of E_i is 4.

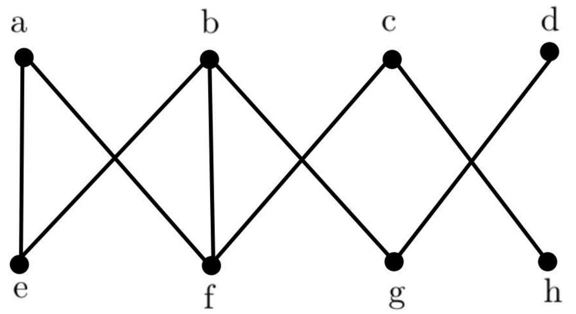

Fig. 12. Unicyclic graph G

H. Cardinality of the edge partition P containing a smallest edge cover

Theorem 11: If one of the edge sets in P is the smallest edge covering set, then P can have at most $m - \lceil \frac{n}{2} \rceil + 1$ edge sets, where n is the order of G and m is the size of G.

Proof: In any graph not having isolated vertices, the sum of the matching number and the edge covering number equals the number of vertices in a graph. Therefore, the size of the smallest edge cover $\beta'(G) = n - |M|$, where n is the number of vertices in a graph and M is a maximum matching. Also, the size of a maximum matching is less than or equal to the edge covering number. Therefore, $|M| \leq \beta'(G)$. That implies, $\beta'(G) \geq \frac{n}{2}$.

Let $P = \{E_1, E_2,...,E_k\}$ be a partition of E and E_i be the smallest edge covering set. Then, $|E_i| \geq \lceil \frac{n}{2} \rceil$ and the remaining edges are taken as singleton sets in P. Then cardinality of P is at most $m - \lceil \frac{n}{2} \rceil + 1$. ■

Corollary 11.1: If E_i in P is a smallest edge covering set and M is a perfect matching, then $|P| = m - \frac{n}{2} + 1$.

Proof: A graph G can contain a perfect matching only when it has an even number of vertices such that all vertices of G are matched. Then, $|M| = \frac{n}{2}$ and $|E_i| = \frac{n}{2}$. This implies, $|P| = m - \frac{n}{2} + 1$. ■

Corollary 11.2: If E_i in P is a smallest edge covering set and M is a near-perfect matching, then $|P| = m - \lceil \frac{n}{2} \rceil + 1$.

Proof: A graph G can contain a near-perfect matching only when it has an odd number of vertices such that only one vertex is unmatched. Then, $|E_i| = \lceil \frac{n}{2} \rceil$. This implies, $|P| = m - \lceil \frac{n}{2} \rceil + 1$. ■

III. Conclusion

This study introduces novel types of generalized complements based on the edge set partitions. These new definitions allow multiple selections of vertices. This unique approach helps to specifically deal with each vertex and its connections. The study outlines the properties, theorems and relationships between the generalized complements concerning edge partitions, contributing to a deeper understanding of graph structures. The characterizations of cycle, path and unicyclic graphs are obtained in this study.

Moreover, k'-complement of a graph, interconnected with a Fibonacci polynomial and maximal outerplanar graph highlights its interplay with fundamental mathematical concepts and its connection to graph theory. Its ability to bridge various fields of study not only enriches our understanding of graph theory but also stimulates innovation and advancements in diverse domains.

Additionally, the comprehensive analysis presented in this research establishes the groundwork for designing novel algorithms and techniques for graph manipulation and optimization. Overall, this study provides a foundation for further research endeavours in this domain.

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