# Roughness Results in Rings

Shreehari Bhaskar, Prakash Padoor and Kavitha Koppula*

***Abstract*—A rough set is an approximation of a subset of a universe. Rough sets are mainly used in decision-making when the given data is uncertain. Rough set theory is a groundbreaking approach that provides a formal framework for extracting facts from imperfect data and helps us classify objects based on their similarities. Developing an algebraic structure for rough sets facilitates a detailed study of the set-theoretic properties. In this paper, we consider the universe as a ring and obtain rough set results. We consider a new equivalence relation on a ring $R$ whose equivalence classes form a partition of $R$. Then, we define upper and lower approximations of a subset of a ring $R$ with respect to the given equivalence relation. Subsequently, we prove related results on these approximations and are illustrated with suitable examples. In addition, we obtain the relationship between the upper and lower approximations defined in this paper and the ones defined earlier.**

***Index Terms*—Ring, Rough set, Ideal, Prime Ideal, Approximations**

## I. Introduction

The issue of imperfect knowledge has been a topic of interest for a long time, attracting the attention of philosophers, logicians, and mathematicians alike. However, over the years, it has emerged as a crucial concern for computer scientists, especially those involved in artificial intelligence. Although there are several methods for comprehending and working with imperfect knowledge, Zadeh's [21] fuzzy set theory stands out as the most effective. Another promising attempt is through rough set theory. Since Pawlak [17] presented his pioneering work, rough set theory has advanced significantly as a tool for information systems modeling and processing of incomplete information. The rough set theory is an extension of the set theory.

Pawlak rough sets are determined using an equivalence relation over a non-empty set called the universe, which is the foundation for establishing upper and lower approximations. Developing an algebraic structure for rough sets facilitates the study of the set-theoretic properties in detail. Different aspects of rough sets have been studied; namely, algebraic properties were examined by Bonikowaski [3], lattice theoretical approach by Iwinski [10], probabilistic approach by Yao [20] and so on. Rough subgroups were first conceptualized by Biswas and Nanda [2]. In Kuroki [15], the concept of a rough ideal in a semigroup was presented, and many characteristics were examined. Mordeson and Kuroki [16] reviewed the structure of rough sets and rough groups. Furthermore, the idea of approximation spaces was extended to the theory of algebraic hyperstructures by Davvaz[5].

Fuzzification of a rough set is a problem that Dubois and Prade [9] started to look into. Fuzzy rough sets were developed to extend the notion of rough sets further. Davvaz [6] proposed a broader perspective on this concept by introducing rough subrings and rough ideals. Later, a rough submodule was introduced as an elaborate idea of a submodule in an R-module [7]. Generalizations of approximation spaces and some extensions of rough sets can be seen in the work by Pawlak [18]. Davvaz [8] defined T-rough sets as a generalization of rough sets where set-valued maps were used instead of equivalence classes. A rough approximation framework (RAF) was proposed by Ciucci [4] as a concept that allowed several approximations on the same set. Subsequently, Kedukodi, Kuncham, and Bhavanari [11] have demonstrated that a RAF can be developed using the idea of reference points. Ali, Davvaz, and Shabir [1] explored the algebraic and topological properties of generalized rough sets corresponding to generalized approximations. Koppula, Kedukodi, and Kuncham [13], [14] proposed a connection between rough sets and Markov chains.

Rough set theory provides a formal framework for extracting facts from imperfect data and helps us classify objects based on their similarities. Sets that cannot be precisely defined are called rough. The boundary set is characterized by objects without definite membership in the set or its complement. The boundary set indicates insufficient knowledge about the set, making it an area with more scope for further research. This innovative concept has the potential to drive significant advancements in data analysis and learning algorithms, making it a fundamental tool in modern-day AI. With its precise mathematical formulation and ability to handle indiscernible objects, rough set theory is a practical approach that can aid in our understanding of complex systems. Rough set theory will be a useful tool in classification theory, cluster

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Shreehari Bhaskar is a Postgraduate student in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (e-mail: shreehari-harry2001@gmail.com).

Prakash Padoor is a Research Scholar in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (e-mail: prakash.padoor@gmail.com).

Kavitha Koppula is an Assistant Professor-Selection Grade in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (*corresponding author to provide phone: 9036417473; e-mail: kavitha.koppula@manipal.edu).

analysis, measurement theory, taxonomy, etc.

In this paper, we discuss some properties of rough sets by considering a different equivalence relation, thereby generalizing the results given by Davvaz [6]. We define a relation $R(K,r)$ on a ring $R$, which partitions $R$ into equivalence classes. Then, we define the upper approximation and lower approximation of a subset of $R$ with respect to $R(K,r)$. We also obtain results on these approximations and illustrate them with appropriate examples.

## II. Preliminaries

In the present section, we provide some basic definitions that are useful to obtain the results of this paper.

**Definition II.1.** A relation $\sim$ on a non empty set $X$ is said to be

1) Reflexive, if $q \sim q$ for all $q \in X$.
2) Symmetric, for $q, n \in X$, if $q \sim n$, then $n \sim q$.
3) Transitive, for $q, n, z \in X$, if $q \sim n$ and $n \sim z$, then $q \sim z$.

A relation that meets each of the aforementioned conditions is called an equivalence relation.

**Definition II.2.** A right (resp. left) ideal $I$ of a ring $R$ is a non-empty subset of $R$ that forms an additive subgroup of $R$ and $e \cdot y \in I, \ \forall\ y \in R,\ e \in I$ (resp. $y \cdot e \in I$). $I$ is an ideal of $R$ if it is both right and left ideal.

**Definition II.3.** An ideal $K$ of a ring $R$ is said to be a prime ideal of $R$, for $f,\ l \in R$, if $f \cdot l \in K$, then either $f \in K$ or $l \in K$.

**Definition II.4.** A non-empty subset $K$ of a ring $R$ is said to be a subring of $R$ if $K$ itself is a ring or, equivalently, if the following conditions hold.

1) $z - y \in K$.
2) $zy \in K$, for all $z, y \in K$.

**Definition II.5.** Let $\emptyset \neq M$ and $\emptyset \neq P$ be any two subsets of a ring $R$. Then the multiplication of $M$ and $P$ is defined as $M \cdot P = \{\sum_{i=1}^{n} m_i \cdot p_i \mid m_i \in M,\ \ p_i \in P\}$,
the addition of $M$ and $P$ is defined as $M + P = \{m + p \mid m \in M,\ p \in P\}$, the subtraction of $M$ and $P$ is defined as $M - P = \{m - p \mid m \in M,\ p \in P\}$ and $M + y = \{m + y \mid m \in M\}$.

**Definition II.6.** The union of all equivalence classes with a non-empty intersection with the set yields an upper approximation of a given set, whereas the collection of all equivalence classes that are proper subsets of the set yields a lower approximation.

**Definition II.7.** A set is considered rough if it's boundary, that is, the difference between the lower and upper approximations, is not empty.

For more results, we refer to [19], [22], [12].

## III. Roughness results

In this section, we show that the relation $R(K,r)$ is an equivalence relation on a ring $R$. Then, we define lower and upper approximations of any subset of a ring $R$ and prove related results. The results are illustrated with suitable examples.
In the following, we consider $R$ as a ring.

**Proposition III.1.** If $K$ is an ideal of a ring $R$, $r \in R$, then

$$R(K,r) = \{(c,d) \in R \times R \mid c \cdot r - d \cdot r \in K\}$$

forms an equivalence relation on $R$.

*Proof:* As $b \cdot r - b \cdot r \in K\ \forall\ b \in R$, we get $(b,b) \in R(K,r)$. Hence $R(K,r)$ is reflexive.
Let $(b,e) \in R(K,r)$. Then $b \cdot r - e \cdot r \in K$.
This implies $-(b \cdot r - e \cdot r) \in K$.
This gives $e \cdot r - b \cdot r \in K$.
Hence $(e,b) \in R(K,r)$. Therefore $R(K,r)$ is symmetric.
Let $(b,e) \in R(K,r)$ and $(e,z) \in R(K,r)$.
Then $b \cdot r - e \cdot r \in K$ and $e \cdot r - z \cdot r \in K$.
This implies $b \cdot r - e \cdot r + e \cdot r - z \cdot r \in K$.
Then $b \cdot r - z \cdot r \in K$.
This gives $(b,z) \in R(K,r)$. Hence $R(K,r)$ is transitive.
∎

**Note 1.** The relation mentioned above is an equivalence relation for any $r \in R$. However, the equivalence classes may be different.
The equivalence relation $R(K,r)$ on $R$ gives rise to a partition of $R$ into equivalence classes. The equivalence class of $q \in R$ with respect to $R(K,r)$ is denoted by $[q]_{(K,r)}$.

$$[q]_{(K,r)} = \{n \in R \mid q \cdot r - n \cdot r \in K\}$$

**Example III.1.** Let $R = \mathbb{Z}_6$ be a ring and $K = \{0,3\}$ be an ideal of $R,\ r = 2$. Then
$[0]_{(K,2)} = \{0,3\}$.
$[1]_{(K,2)} = \{1,4\}$.
$[2]_{(K,2)} = \{2,5\}$.

**Note 2.** If $r = 1$, then $[q]_{(K,r)} = [q]_{(K,1)} = [q]_K = \{n \in R \mid q - n \in K\}$.

**Definition III.1.** Let $X$ be a subset of $R$. Then the lower approximation of a set $X$ is the set of all elements in $R$ such that their equivalence class is entirely contained in $X$. That is,

$$\underline{Appr}_{(I,r)}(X) = \{k \in R \mid [k]_{(I,r)} \subseteq X\}.$$

**Definition III.2.** Let $X$ be a subset of $R$. Then the upper approximation of a set $X$ is the set of all elements in $R$ such that their equivalence class makes a non-empty intersection with $X$. That is,

$$\overline{Appr}_{(I,r)}(X) = \{k \in R \mid [k]_{(I,r)} \cap X \neq \emptyset\}.$$

Boundary of $X$ is given by,

$$BND(X) = \overline{Appr}_{(I,r)}(X) - \underline{Appr}_{(I,r)}(X).$$

The set $X$ is called a **Rough set** if the boundary of $X$ is non-empty. Otherwise, $X$ is called **Crisp**.

**Example III.2.** Let $R = \mathbb{Z} \times \mathbb{Z}$. Then $R$ is a ring with respect to addition and multiplication defined as below.

$$(e, f) + (g, h) = (e + g, f + h)$$

$$(e, f) \cdot (g, h) = (e \cdot g, f \cdot h),\ \forall\ (e, f), (g, h) \in \mathbb{Z} \times \mathbb{Z}$$

Then $K = 4\mathbb{Z}\times 4\mathbb{Z}$ is an ideal of $R$ and $r = (1, 2) \in \mathbb{Z}\times\mathbb{Z}$. The equivalence relation $R(K, r)$ partitions $R$ into the equivalence classes as,

$$\begin{aligned}[(0, 0)]_{(K,r)} &= \{(e', g') \mid (-e', -2g') \in 4\mathbb{Z} \times 4\mathbb{Z}\} \\ &= \{(e', g') \mid e' \in 4\mathbb{Z}, g' \in 2\mathbb{Z}\}\end{aligned}$$

$[(1, 1)]_{(K,r)} = \{(e', g') \mid e' \in 4\mathbb{Z} + 1, g' \notin 2\mathbb{Z}\}$
$[(2, 2)]_{(K,r)} = \{(e', g') \mid e' \in 4\mathbb{Z} + 2, g' \in 2\mathbb{Z}\}$
$[(3, 3)]_{(K,r)} = \{(e', g') \mid e' \in 4\mathbb{Z} + 3, g' \notin 2\mathbb{Z}\}$
Let $A = \{(e, g) \mid e \in (5\mathbb{Z} \cup 4\mathbb{Z}), g \in 2\mathbb{Z}\}$. Then

$$\underline{Appr}_{(K,r)}(A) = \{(e, g) \mid e \in 4\mathbb{Z}, g \in 2\mathbb{Z}\}$$

$$\begin{aligned}\overline{Appr}_{(K,r)}(A) &= \{(e, g) \mid [(e, g)]_{(K,r)} \cap A \neq \emptyset\} \\ &= \{(e, g) \mid e \in 2\mathbb{Z}, g \in 2\mathbb{Z}\}\end{aligned}$$

$$\begin{aligned}BND(A) &= \overline{Appr}_{(K,r)}(A) - \underline{Appr}_{(K,r)}(A) \\ &= \{(e, g) \mid e \in 4\mathbb{Z} + 2, g \in 2\mathbb{Z}\} \neq \emptyset\end{aligned}$$

Hence $A$ is a rough set.
Let $L = 2\mathbb{Z} \times 2\mathbb{Z}$. Then
$\overline{Appr}_{(K,r)}(L) = \{(e, g) \mid [(e, g)]_{(I,r)} \cap L \neq \emptyset\} = 2\mathbb{Z} \times 2\mathbb{Z}$
$\underline{Appr}_{(K,r)}(L) = \{x \in \mathbb{Z} \times \mathbb{Z} \mid [x]_{(K,r)} \subseteq L\} = 2\mathbb{Z} \times 2\mathbb{Z}$
$BND(L) = \overline{Appr}_{(K,r)}(L) - \underline{Appr}_{(K,r)}(L) = \emptyset$
Hence $L$ is crisp set.

**Proposition III.2.** If $\emptyset \neq Z$ and $\emptyset \neq Y$ are any two subsets of $R$, $r \in R$ and $(R, R(K, r))$ is an approximation space, then we have,

1) $\underline{Appr}_{(K,r)}(Z) \subseteq Z \subseteq \overline{Appr}_{(K,r)}(Z)$.
2) $\underline{Appr}_{(K,r)}(\emptyset) = \emptyset = \overline{Appr}_{(K,r)}(\emptyset)$.
3) $\underline{Appr}_{(K,r)}(R) = R = \overline{Appr}_{(K,r)}(R)$.
4) If $Z \subseteq Y$, then $\underline{Appr}_{(K,r)}(Z) \subseteq \underline{Appr}_{(K,r)}(Y)$ and $\overline{Appr}_{(K,r)}(Z) \subseteq \overline{Appr}_{(K,r)}(Y)$.
5) $\underline{Appr}_{(K,r)}(Z) = (\overline{Appr}_{(K,r)}(Z^C))^C$.
6) $\overline{Appr}_{(K,r)}(Z) = (\underline{Appr}_{(K,r)}(Z^C))^C$.
7) $\underline{Appr}_{(K,r)}(Z \cap Y) = \underline{Appr}_{(K,r)}(Z) \cap \underline{Appr}_{(K,r)}(Y)$.
8) $\overline{Appr}_{(K,r)}(Z \cap Y) \subseteq \overline{Appr}_{(K,r)}(Z) \cap \overline{Appr}_{(K,r)}(Y)$.
9) $\underline{Appr}_{(K,r)}(Z \cup Y) \supseteq \underline{Appr}_{(K,r)}(Z) \cup \underline{Appr}_{(K,r)}(Y)$.
10) $\overline{Appr}_{(K,r)}(Z \cup Y) = \overline{Appr}_{(K,r)}(Z) \cup \overline{Appr}_{(K,r)}(Y)$.
11) $\underline{Appr}_{(K,r)}([a]_{(K,r)}) = \overline{Appr}_{(K,r)}([a]_{(K,r)})\ \forall\ a \in R$.

Now, we provide an example to demonstrate that the converse of 8 and 9 of the Proposition III.2 is not necessarily true in general.

**Example III.3.** Let $R = \mathbb{Z}_8$ be a ring and $K = \{0, 2, 4, 6\}$ be an ideal of $\mathbb{Z}_8$.
The equivalence relation $R(K, 3)$ partitions $R$ into the equivalence classes as,
$[0]_{(K,3)} = [2]_{(K,3)} = [4]_{(K,3)} = [6]_{(K,3)} = \{0, 2, 4, 6\}$
$[1]_{(K,3)} = [3]_{(K,3)} = [5]_{(K,3)} = [7]_{(K,3)} = \{1, 3, 5, 7\}$
Now, take $X = \{0, 2, 4\}$, $Y = \{6\}$. Then
$\overline{Appr}_{(K,3)}(X) = \{0, 2, 4, 6\}$, $\overline{Appr}_{(K,3)}(Y) = \{0, 2, 4, 6\}$
Now, $X \cap Y = \emptyset$. Then $\overline{Appr}_{(K,3)}(X \cap Y) = \emptyset$,
$\overline{Appr}_{(K,3)}(Y) \cap \overline{Appr}_{(K,3)}(X) = \{0, 2, 4, 6\}$,
$\underline{Appr}_{(I,3)}(X) = \emptyset$, $\underline{Appr}_{(K,3)}(Y) = \emptyset$.
Now, $X \cup Y = \{0, 2, 4, 6\}$.
Then $\underline{Appr}_{(I,3)}(X \cup Y) = \{0, 2, 4, 6\}$,
$\underline{Appr}_{(K,3)}(Y) \cup \underline{Appr}_{(K,3)}(X) = \emptyset$, and so
$(\overline{Appr}_{(K,3)}(X) \cap \overline{Appr}_{(K,3)}(Y)) \nsubseteq \overline{Appr}_{(K,3)}(X \cap Y)$ and
$\underline{Appr}_{(K,3)}(X \cup Y) \nsubseteq (\underline{Appr}_{(K,3)}(X) \cup \underline{Appr}_{(K,3)}(Y))$.

**Proposition III.3.** If $K$ and $L$ are any two ideals of $R$, $r \in R$, then $R(K \cap L, r) = R(K, r) \cap R(L, r)$.

*Proof:* Let $(c, d) \in R(K \cap L, r)$. Then $c \cdot r - d \cdot r \in (K \cap L)$. This implies $c \cdot r - d \cdot r \in K$ and $c \cdot r - d \cdot r \in L$.
This gives $(c, d) \in R(K, r)$ and $(c, d) \in R(L, r)$.
Then $(c, d) \in R(K, r) \cap R(L, r)$.
Hence we get $R(K \cap L, r) \subseteq R(K, r) \cap R(L, r)$.
Conversely, let $(c, d) \in R(K, r) \cap R(L, r)$.
Then $c \cdot r - d \cdot r \in K$ and $c \cdot r - d \cdot r \in L$.
This implies $c{\cdot}r-d{\cdot}r \in (K\cap L)$. Hence $(c, d) \in R(K\cap L, r)$.
Therefore $R(K, r) \cap R(K, r) \subseteq R(K \cap L, r)$.
Thus $R(K \cap L, r) = R(K, r) \cap R(K, r)$. ■

**Proposition III.4.** If $K$ is a right ideal of $R$, $r \in R$, then $[c]_K \subseteq [c]_{(K,r)}$ for all $c \in R$ .

*Proof:* Let $d \in [c]_K$. Then $c - d \in K$. As $r \in R$, we get $(c - d) \cdot r \in K$.
This implies $c \cdot r - d \cdot r \in K$. This gives $d \in [c]_{(K,r)}$.
Hence $[c]_K \subseteq [c]_{(K,r)}$. ■

If $K$ is a left ideal of $R$, then the Proposition III.4 is not necessarily true.

**Example III.4.** Let $R = M_{2\times 2}(\mathbb{Z})$ and $K = \left\{\begin{pmatrix} a & 2b \\ c & 2d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\right\}$ be a left ideal of $R$.

Now, take $r = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_{\left(K, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)} =$
$\left\{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) \mid \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in K\right\}$
$= \left\{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) \mid \begin{pmatrix} e+f & e+f \\ g+h & g+h \end{pmatrix} \in K\right\}$
$\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_K = \left\{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) \mid \begin{pmatrix} e & f \\ g & h \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in K\right\}$
$= K$

We have $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_K$.
But $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix} \notin K$.
This implies $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \notin \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_{\left(K, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)}$.
Hence $[c]_K \nsubseteq [c]_{(K,r)}$.

**Proposition III.5.** If $K$ is a right ideal of $R$ and $K$ is prime, $r \in R$ and $r \notin K$, then $[c']_K = [c']_{(K,r)}$, for all $c' \in R$.

*Proof:* Let $d \in [c']_{(K,r)}$. Then $c' \cdot r - d \cdot r \in K$.
This implies $(c' - d) \cdot r \in K$.
As $r \notin K$ and $K$ is prime, we get $c' - d \in K$.
This gives $d \in [c']_K$. Hence $[c']_{(K,r)} \subseteq [c']_K$.
By Proposition III.4, we have $[c']_K \subseteq [c']_{(K,r)}$.
Thus $[c']_K = [c']_{(K,r)}$. ■

In the below example, we can observe that if $K$ is not prime, then the above result need not hold in general.

**Example III.5.** Let $R = \mathbb{Z}$ and $K = 4\mathbb{Z}$ be an ideal of $\mathbb{Z}$, which is not prime.
Then the equivalence relation $R(K)$ partitions $\mathbb{Z}$ as,
$[0]_K = \{0, \pm 4, \pm 8, \pm 12, \cdots\}$,
$[1]_K = \{1, -3, 5, -7, 9, -11, 13, -15, \cdots\}$,
$[2]_K = \{\pm 2, \pm 6, \pm 10, \cdots\}$,
$[3]_K = \{-1, 3, -5, 7, -9, 11, -13, 15, \cdots\}$.
Let $r = 2$. The equivalence relation $R(K, 2)$ partitions $\mathbb{Z}$ as,
$[0]_{(K,2)} = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \cdots\}$,
$[1]_{(K,2)} = \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \cdots\}$.
We can observe that $[0]_{(K,2)} \nsubseteq [0]_K$.

**Proposition III.6.** If $K$ is an ideal of $R$ and $q, d, r \in R$, then $[q - d]_{(K,r)} = [q]_{(K,r)} - [d]_{(K,r)}$.

*Proof:* Let $z \in [q - d]_{(K,r)}$. Then $z \cdot r - (q - d) \cdot r \in K$.
This implies $z \cdot r + d \cdot r - q \cdot r \in K$.
Then $(z + d) \cdot r - q \cdot r \in K$. This gives $z + d \in [q]_{(K,r)}$.
Then $z \in ([q]_{(K,r)} - d) \subseteq [q]_{(K,r)} - [d]_{(K,r)}$.
This implies $z \in [q]_{(K,r)} - [d]_{(K,r)}$.
Hence $[q - d]_{(K,r)} \subseteq [q]_{(K,r)} - [d]_{(K,r)}$.
Conversely, let $z \in [q]_{(K,r)} - [d]_{(K,r)}$. Then $z = z' - z''$ for some $z' \in [q]_{(K,r)}$ and $z'' \in [d]_{(K,r)}$.
This implies $z' \cdot r - q \cdot r \in K$ and $z'' \cdot r - d \cdot r \in K$.
Then $z' \cdot r - q \cdot r \in K$ and $-(z'' \cdot r - d \cdot r) \in K$. Now, we get $z' \cdot r - q \cdot r - (z'' \cdot r - d \cdot r) \in K$ (closure property of $K$). This implies $(z' - z'') \cdot r - (q - d) \cdot r \in K$.
$(z' - z'') \in [q - d]_{(K,r)}$. Hence $z \in [q - d]_{(K,r)}$.
Therefore $[q]_{(K,r)} - [d]_{(K,r)} \subseteq [q - d]_{(K,r)}$.
Thus $[q - d]_{(K,r)} = [q]_{(K,r)} - [d]_{(K,r)}$. ■

**Proposition III.7.** If $K$ and $L$ are ideals of $R$ and $q, d, r \in R$, then $[q]_{(K,r)} + [d]_{(L,r)} \subseteq [q + d]_{(K+L,r)}$.

*Proof:* Let $u \in [q]_{(K,r)} + [d]_{(L,r)}$. Then $u = e' + l'$ for some $e' \in [q]_{(K,r)}$ and $n' \in [d]_{(L,r)}$.
This implies $e' \cdot r - q \cdot r \in K$ and $n' \cdot r - d \cdot r \in L$.
Then $(e' \cdot r - q \cdot r) + (n' \cdot r - d \cdot r) = (e' + n') \cdot r - (q + d) \cdot r \in (K + L)$.
This implies $(e' + n') = u \in [q + d]_{(K+L,r)}$.
Hence $[q]_{(K,r)} + [d]_{(L,r)} \subseteq [q + d]_{(K+L,r)}$. ■

**Proposition III.8.** If $K$ and $L$ are ideals of $R$ together with $K \subseteq L$, $r \in R$, then $[c]_{(K,r)} \subseteq [c]_{(L,r)}$, for all $c \in R$.

*Proof:* Let $d \in [c]_{(K,r)}$. Then $d \cdot r - c \cdot r \in K$. This implies $d \cdot r - c \cdot r \in J$ (since $K \subseteq L$). Then $d \in [c]_{(L,r)}$.
Hence $[c]_{(K,r)} \subseteq [c]_{(L,r)}$. ■

**Proposition III.9.** If $K$ is an ideal of $R$, $r \in R$, then $c + [0]_{(K,r)} = [c]_{(K,r)}$, for all $c \in R$.

*Proof:* Let $d \in c + [0]_{(K,r)}$. Then $d = c + z$ for some $z \in [0]_{(K,r)}$. This implies $z \cdot r - 0 \cdot r \in K$. Then $z \cdot r \in K$.
This gives $(d - c) \cdot r \in K$. Then $d \cdot r - c \cdot r \in K$.
This implies $d \in [c]_{(K,r)}$. Hence $c + [0]_{(K,r)} \subseteq [c]_{(K,r)}$.
Conversely, let $d \in [c]_{(K,r)}$. Then $d \cdot r - c \cdot r \in K$.
This implies $(d - c) \cdot r - 0 \cdot r \in K$. This gives $(d - c) \in [0]_{(K,r)}$.
Then $d \in [0]_{(K,r)} + c$. Hence $[c]_{(K,r)} \subseteq [0]_{(K,r)} + c$.
Thus $c + [0]_{(K,r)} = [c]_{(K,r)}$. ■

**Proposition III.10.** If $K$ is left ideal of $R$, $r \in R$, then $[0]_{(K,r)}$ is a left ideal of $R$.

*Proof:* Let $c, d \in [0]_{(K,r)}$. Then $c \cdot r \in K$ and $d \cdot r \in K$.
This implies $(c - d) \cdot r \in K$.
Then $c \cdot r - d \cdot r \in K$, which gives $(c - d) \in [0]_{(K,r)}$.
Now, take $c \in [0]_{(K,r)}$ and $a \in R$. Then $c \cdot r \in K$ and $a \in R$.
As $K$ is a left ideal of $R$, we get $a \cdot (c \cdot r) \in K$. This implies $a \cdot c \in [0]_{(K,r)}$. Hence $[0]_{(K,r)}$ is a left ideal of $R$. ■

**Proposition III.11.** If $K$ is an ideal of a commutative ring $R$, $c, d, r \in R$, then $([c]_{(K,r)}) \cdot ([d]_{(K,r)}) \subseteq [(c \cdot d)]_{(K,r)}$.

*Proof:* Now, we show that $([0]_{(K,r)} + c) \cdot ([0]_{(K,r)} + d) \subseteq [0]_{(K,r)} + (c \cdot d)$ (By Proposition III.9). Consider $([0]_{(K,r)} + c) \cdot ([0]_{(K,r)} + d) = ([0]_{(K,r)}) \cdot ([0]_{(K,r)}) + ([0]_{(K,r)}) \cdot d + c \cdot ([0]_{(K,r)}) + c \cdot d$ $([0]_{(K,r)}) \cdot ([0]_{(K,r)}) = \{\sum_{i=1}^{n} (a_i) \cdot (b_i) \mid a_i, b_i \in [0]_{(K,r)}\} \subseteq [0]_{(K,r)}$ (As $[0]_{(K,r)}$ is an ideal of a ring $R$).
Similarly $(c \cdot [0]_{(K,r)}) \subseteq [0]_{(K,r)}$ and $([0]_{(K,r)} \cdot d) \subseteq [0]_{(K,r)}$.
Hence $([0]_{(K,r)} + c) \cdot ([0]_{(K,r)} + d) \subseteq [0]_{(K,r)} + (c \cdot d)$.
Therefore $([c]_{(K,r)}) \cdot ([d]_{(K,r)}) \subseteq [(c \cdot d)]_{(K,r)}$. ■

**Proposition III.12.** If $K$ is a right ideal of $R$ and $\emptyset \neq A$ is any subset of $R$, $r \in R$, then $\overline{Appr}_K(A) \subseteq \overline{Appr}_{(K,r)}(A)$ and $\underline{Appr}_{(K,r)}(A) \subseteq \underline{Appr}_K(A)$.

*Proof:* Let $c \in \overline{Appr}_K(A)$. Then $[c]_K \cap A \neq \emptyset$.
This implies $d \in ([c]_K \cap A)$ for some $d \in R$.
Then $d \in [c]_K$ and $d \in A$.
As $K$ is a right ideal of $R$, then by Proposition III.4, we have $[c]_K \subseteq [c]_{(K,r)}$. This implies $d \in [c]_K \subseteq [c]_{(K,r)}$.
This gives $[c]_{(K,r)} \cap A \neq \emptyset$. Then $c \in \overline{Appr}_{(K,r)}(A)$.
Hence $\overline{Appr}_K(A) \subseteq \overline{Appr}_{(K,r)}(A)$.
Let $c \in \underline{Appr}_{(K,r)}(A)$. Then $[c]_{(K,r)} \subseteq A$.
Since $K$ is right ideal of $R$, we have $[c]_K \subseteq [c]_{(K,r)}$.
Hence $[c]_K \subseteq A$. Therefore $c \in \underline{Appr}_K(A)$.
Thus $\underline{Appr}_{(K,r)}(A) \subseteq \underline{Appr}_K(A)$. ■

**Corollary III.0.1.** If $K$ is right ideal of $R$ and $A$ is any subset of $R$, $r \in R$, then
$\underline{Appr}_{(K,r)}(A) \subseteq \underline{Appr}_K(A) \subseteq A \subseteq \overline{Appr}_K(A) \subseteq \overline{Appr}_{(K,r)}(A)$.

**Proposition III.13.** If $K$ is left ideal of $R$ and $r \in K$, then $[c]_{(K,r)} = R$, for all $c \in R$.

*Proof:* We know that $[c]_{(K,r)} \subseteq R$, for all $c \in R$.
As $c \in R$ and $r \in K$, we have $c \cdot r \in K$.
Now, take $y \in R$. As $r \in K$ and $K$ is a left ideal of $R$, we have $y \cdot r \in K$.
Then $c \cdot r - y \cdot r \in K$. This implies $y \in [c]_{(K,r)}$.
Then $R \subseteq [c]_{(K,r)}$. Hence $[c]_{(K,r)} = R$, $\forall\, c \in R$. ■

**Proposition III.14.** If $R$ is a ring and $L \subseteq R$, $K$ is a left ideal of $R$ and $r \in K$, then $\overline{Appr}_{(K,r)}(L) = \begin{cases} R, & \text{if } L \neq \emptyset \\ \emptyset, & \text{if } L = \emptyset \end{cases}$ and $\underline{Appr}_{(K,r)}(L) = \begin{cases} \emptyset, & \text{if } L \subset R \\ R, & \text{if } L = R. \end{cases}$

*Proof:* The proof is straight forward by Proposition III.13.

$$\overline{Appr}_{(K,r)}(L) = \{q \in R \mid [q]_{(K,r)} \cap L \neq \emptyset\} = \begin{cases} R, & \text{if } L \neq \emptyset \\ \emptyset, & \text{if } L = \emptyset \end{cases}$$

$$\underline{Appr}_{(K,r)}(L) = \{q \in R \mid [q]_{(K,r)} \subseteq L\} = \begin{cases} \emptyset \text{ if } L \subset R \\ R \text{ if } L = R \end{cases}$$
■

**Proposition III.15.** If $K$ and $I$ are ideals of $R$ with $I \subseteq K$ and $\emptyset \neq L \subseteq R$, $r \in R$, then

1) $\underline{Appr}_{(K,r)}(L) \subseteq \underline{Appr}_{(I,r)}(L)$
2) $\overline{Appr}_{(I,r)}(L) \subseteq \overline{Appr}_{(K,r)}(L)$

*Proof:* 1. Let $q \in \underline{Appr}_{(K,r)}(L)$. Then $[q]_{(K,r)} \subseteq L$.
This implies $[q]_{(I,r)} \subseteq L$ (By Proposition III.8 we have $[q]_{(I,r)} \subseteq [q]_{(K,r)}$). This gives $q \in \underline{Appr}_{(I,r)}(L)$.
Hence $\underline{Appr}_{(K,r)}(L) \subseteq \underline{Appr}_{(I,r)}(L)$.
2. Let $q \in \overline{Appr}_{(I,r)}(L)$. Then $[q]_{(I,r)} \cap L \neq \emptyset$.
This implies $[q]_{(K,r)} \cap L \neq \emptyset$ (As $[q]_{(I,r)} \subseteq [q]_{(K,r)}$).
Hence $q \in \overline{Appr}_{(K,r)}(L)$.
Therefore $\overline{Appr}_{(I,r)}(L) \subseteq \overline{Appr}_{(K,r)}(L)$. ■

**Corollary III.0.2.** If $T$ and $K$ are ideals of $R$ and $\emptyset \neq P$ is any subset of $R$, $r \in R$, then

1) $\underline{Appr}_{(T,r)}(P) \cap \underline{Appr}_{(K,r)}(P) \subseteq \underline{Appr}_{(T\cap K,r)}(P)$.
2) $\overline{Appr}_{(T\cap K,r)}(P) \subseteq \overline{Appr}_{(T,r)}(P) \cap \overline{Appr}_{(K,r)}(P)$.

*Proof:* This can proved by the fact that $T \cap K \subseteq T$ and $T \cap K \subseteq K$ and by using the Proposition III.15. ■

Here, we give an example to show that the converse of the Proposition III.15 is not necessarily valid in general.

**Example III.6.** Let $R = \mathbb{Z}_6$ and the ideals of $\mathbb{Z}_6$ are $I = \{0\}$, $K = \{0, 2, 4\}$. Now, $A = \{0, 1\}$ and $B = \{2\}$ are subsets of $\mathbb{Z}_6$ and $r = 5$.
The equivalence relation $R(I, 5)$ partitions $\mathbb{Z}_6$ into the equivalence classes as,
$[0]_{(I,5)} = \{0\}$, $[1]_{(I,5)} = \{1\}$, $[2]_{(I,5)} = \{2\}$,
$[3]_{(I,5)} = \{3\}$ , $[4]_{(I,5)} = \{4\}$, $[5]_{(I,5)} = \{5\}$.
The equivalence relation $R(K, 5)$ partitions $\mathbb{Z}_6$ into the equivalence classes as,
$[0]_{(K,5)} = [2]_{(K,5)} = [4]_{(K,5)} = \{0, 2, 4\}$,
$[1]_{(K,5)} = [3]_{(K,5)} = [5]_{(K,5)} = \{1, 3, 5\}$.
Here $I \subset K$. $\underline{Appr}_{(I,5)}(A) = \{0, 1\}$ and $\underline{Appr}_{(K,5)}(A) = \emptyset$.
Hence $\underline{Appr}_{(I,5)}(A) \nsubseteq \underline{Appr}_{(K,5)}(A)$.
$\overline{Appr}_{(I,5)}(B) = \{2\}$ and $\overline{Appr}_{(K,5)}(B) = \{0, 2, 4\}$.
Hence $\overline{Appr}_{(K,5)}(B) \nsubseteq \overline{Appr}_{(I,5)}(B)$.

**Proposition III.16.** If $K$ is right ideal of $R$ and $K$ is prime, $\emptyset \neq P$ and $\emptyset \neq S$ are any two subsets of $R$, $r \in R$, $r \notin K$ and $K^2 = K$, then $\overline{Appr}_{(K,r)}(P \cdot S) = \overline{Appr}_{(K,r)}(P) \cdot \overline{Appr}_{(K,r)}(S)$.

In the following example, we can observe that if $K$ is a right ideal of $R$ and is not prime, then $(\overline{Appr}_{(K,r)}P) \cdot (\overline{Appr}_{(K,r)}S) = \overline{Appr}_{(K,r)}(P \cdot S)$ need not be true.

**Example III.7.** Let $2\mathbb{Z}$ be a ring and $4\mathbb{Z}$ be an ideal of $2\mathbb{Z}$. Now, take the subsets $P = \{2\}$, $S = \{4\}$ of $2\mathbb{Z}$ and $r = 4$.
Then $[x]_{(I,4)} = \{y \in 2\mathbb{Z} \mid x \cdot 4 - y \cdot 4 \in 4\mathbb{Z}\} = \{y \in 2\mathbb{Z} \mid (x - y) \cdot 4 \in 4\mathbb{Z}\} = 2\mathbb{Z}\ \forall\, x \in 2\mathbb{Z}$.
$\overline{Appr}_{(4\mathbb{Z},4)}(P) = \{x \in 2\mathbb{Z} \mid [x]_{(4\mathbb{Z},4)} \cap A \neq \emptyset\} = 2\mathbb{Z}$. (As $4 \in 4\mathbb{Z}$ and $4\mathbb{Z}$ is a left ideal of $\mathbb{Z}$, by Proposition III.14 ).
Similarly, $\overline{Appr}_{(4\mathbb{Z},4)}(S) = 2\mathbb{Z}$. ( As $4 \in 4\mathbb{Z}$ and $4\mathbb{Z}$ is a left ideal of $\mathbb{Z}$, by Proposition III.14 ).
Now, $P \cdot S = \{8, 16, 32, 40, \cdots, (8 \cdot n)\}$.
Then $\overline{Appr}_{(4\mathbb{Z},4)}(P \cdot S) = 2\mathbb{Z}$.
But $(\overline{Appr}_{(4\mathbb{Z},4)}P) \cdot (\overline{Appr}_{(4\mathbb{Z},4)}S) = 2\mathbb{Z} \cdot 2\mathbb{Z}$.
We can observe that $\overline{Appr}_{(4\mathbb{Z},4)}(P \cdot S) \nsubseteq (\overline{Appr}_{(4\mathbb{Z},4)}P) \cdot (\overline{Appr}_{(4\mathbb{Z},4)}S)$ ($2 \in 2\mathbb{Z}$, but $2 \notin 2\mathbb{Z} \cdot 2\mathbb{Z}$).

The below example shows that

$(\overline{Appr}_{(K,r)}P) \cdot (\overline{Appr}_{(K,r)}S) \nsubseteq \overline{Appr}_{(K,r)}(P \cdot S)$, when $K$ is not a left ideal of $R$.

**Example III.8.** Let $R = M_{2\times2}(\mathbb{Z})$ and $K = \left\{\begin{pmatrix} k & l \\ 0 & 0 \end{pmatrix} \mid k, l \in \mathbb{Z}\right\}$ is a right ideal of $R$. Now, take $P = \left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ and $S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ are subsets of $R$ and $r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2\times2}(\mathbb{Z})$.

Then the equivalence class of any matrix is $\left[\begin{pmatrix} p & q \\ s & t \end{pmatrix}\right]_{\left(K, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)} =$

$$\left\{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times2}(\mathbb{Z}) \mid \begin{pmatrix} p & q \\ s & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in K\right\}$$

$$= \left\{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times2}(\mathbb{Z}) \mid \begin{pmatrix} p & 0 \\ s & 0 \end{pmatrix} - \begin{pmatrix} e & 0 \\ g & 0 \end{pmatrix} \in K\right\}$$

$$= \left\{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times2}(\mathbb{Z}) \mid e, f, g, h \in \mathbb{Z} \text{ and } s = g\right\}.$$

Now, $\overline{Appr}_{(K,r)}P = \{P' \in M_{2\times2}(\mathbb{Z}) \mid [P']_{(K,r)} \cap P \neq \emptyset\}$

$$= \left\{\begin{pmatrix} p & q \\ s & t \end{pmatrix} \in M_{2\times2}(\mathbb{Z}) \mid \left[\begin{pmatrix} p & q \\ s & t \end{pmatrix}\right]_{\left(K, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)} \cap P \neq \emptyset\right\}$$

$$= \left\{\begin{pmatrix} p & q \\ s & t \end{pmatrix} \in M_{2\times2}(\mathbb{Z}) \mid \left[\begin{pmatrix} p & q \\ s & t \end{pmatrix}\right]_{\left(K, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)} \cap \left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right\} \neq \emptyset\right\}$$

$$= \left\{\begin{pmatrix} p & q \\ 1 & t \end{pmatrix} \mid p, q, t \in \mathbb{Z}\right\}$$

Similarly, $\overline{Appr}_{(K,r)}(S) = \left\{\begin{pmatrix} p & q \\ 0 & t \end{pmatrix} \mid p, q, t \in \mathbb{Z}\right\}$

$$P \cdot S = \left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} n & 0 \\ n & 0 \end{pmatrix}\right\}.$$

This implies $\overline{Appr}_{(K,r)}(P \cdot S) = \{P' \in M_{2\times2}(\mathbb{Z}) \mid [P']_{(K,r)} \cap (P \cdot S) \neq \emptyset\}$

$$= \left\{\begin{pmatrix} k & l \\ u & v \end{pmatrix} \mid k, l, v \in \mathbb{Z} \text{ and } u = 1, 2, 3, 4, \cdots\right\}.$$

Then $(\overline{Appr}_{(K,r)}P) \cdot (\overline{Appr}_{(K,r)}S) = \{\sum a_i \cdot b_i \mid a_i \in \overline{Appr}_{(K,r)}P \text{ and } b_i \in \overline{Appr}_{(K,r)}S\}$.

Now, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \overline{Appr}_{(K,r)}(P)$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \overline{Appr}_{(K,r)}(S)$.

But $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin \overline{Appr}_{(k,r)}(P \cdot S)$.

Hence $(\overline{Appr}_{(K,r)}P) \cdot (\overline{Appr}_{(K,r)}S) \nsubseteq \overline{Appr}_{(K,r)}(P \cdot S)$.

**Proposition III.17.** If $K$ is an ideal of $R$ and $T \neq \emptyset, L \neq \emptyset$ are any two subsets of $R$, $r \in R$, then $\overline{Appr}_{(K,r)}(T) + \overline{Appr}_{(K,r)}(L) = \overline{Appr}_{(K,r)}(T + L)$.

*Proof:* Let $z \in \overline{Appr}_{(K,r)}(T) + \overline{Appr}_{(K,r)}(L)$. Then $z = q + t$ for some $q \in \overline{Appr}_{(K,r)}(T)$ and $t \in \overline{Appr}_{(K,r)}(L)$.
This implies $[q]_{(K,r)} \cap T \neq \emptyset$ and $[t]_{(K,r)} \cap L \neq \emptyset$.
Now, take $q' \in [q]_{(K,r)} \cap T$ and $t' \in [t]_{(K,r)} \cap L$. Then $q' \in [q]_{(K,r)}$ and $q' \in T$, $t' \in [t]_{(K,r)}$ and $t' \in L$.
This implies $(q' + t') \in ([q]_{(K,r)} + [t]_{(K,r)}) = [q + t]_{(K,r)}$ (By proposition III.6) and $(q' + t') \in (T + L)$.
Then $(q' + t') \in [q + t]_{(K,r)} \cap (T + L)$.
This gives $[q + t]_{(K,r)} \cap (T + L) \neq \emptyset$.
$\implies q + t \in \overline{Appr}_{(K,r)}(T + L)$.
Hence, we get $z \in \overline{Appr}_{(K,r)}(T + L)$.
Thus $\overline{Appr}_{(K,r)}(T) + \overline{Appr}_{(K,r)}(L) \subseteq \overline{Appr}_{(K,r)}(T + L)$.
Let $z \in \overline{Appr}_{(K,r)}(T + L)$. Then $[z]_{(K,r)} \cap (T + L) \neq \emptyset$.
Now, take $t \in [z]_{(K,r)} \cap (T + L)$.
Then $t \in [z]_{(K,r)}$ and $t \in (T + L)$.
This implies $t = a' + b'$ for some $a' \in T$ and $b' \in L$ and $t \in [z]_{(K,r)}$.
We know that $[z]_{(K,r)} = [t]_{(K,r)} = [a' + b']_{(K,r)} = [a']_{(K,r)} + [b']_{(K,r)}$.
As $z \in [z]_{(K,r)} = [a']_{(K,r)} + [b']_{(K,r)}$, we get $z = u + v$ for some $u \in [a']_{(K,r)}$ and $v \in [b']_{(K,r)}$.
As $u \in [a']_{(K,r)}$, we get $[a']_{(K,r)} = [u]_{(K,r)}$ and since $v \in [b']_{(K,r)}$, we get $[b']_{(K,r)} = [v]_{(K,r)}$.
This implies $[u]_{(K,r)} \cap T \neq \emptyset$ and $[v]_{(K,r)} \cap L \neq \emptyset$. Then $u \in \overline{Appr}_{(K,r)}(T)$ and $v \in \overline{Appr}_{(K,r)}(L)$.
Hence $z = u + v \in \overline{Appr}_{(K,r)}(T) + \overline{Appr}_{(K,r)}(L)$.
Therefore $\overline{Appr}_{(K,r)}(T + L) \subseteq \overline{Appr}_{(K,r)}(T) + \overline{Appr}_{(K,r)}(L)$.
Thus $\overline{Appr}_{(K,r)}(T) + \overline{Appr}_{(K,r)}(L) = \overline{Appr}_{(K,r)}(T + L)$. ■

**Proposition III.18.** If $K$ and $T$ are left ideals of $R$, $r \in R$, then $\overline{Appr}_{(K,r)}(T)$ is a left ideal of $R$.

*Proof:* Let $q, t \in \overline{Appr}_{(K,r)}(T)$. Then $[q]_{(K,r)} \cap T \neq \emptyset$ and $[t]_{(K,r)} \cap T \neq \emptyset$.
Now, take $z' \in [q]_{(K,r)} \cap T$ and $z'' \in [t]_{(K,r)} \cap T$.
This implies $z' \in [q]_{(K,r)}$ and $z' \in T$, $z'' \in [t]_{(K,r)}$ and $z'' \in T$.
Then $(z' - z'') \in T$ (As $T$ is an ideal of $R$) and $(z' - z'') \in ([q]_{(K,r)} - [t]_{(K,r)})$.
Now, by Proposition III.6, we have $[q - t]_{(K,r)} = [q]_{(K,r)} - [t]_{(K,r)}$.
This implies $(z' - z'') \in [q - t]_{(K,r)}$. This gives $(z' - z'') \in ([q - t]_{(K,r)} \cap T)$.
Then $([q - t]_{(K,r)} \cap T) \neq \emptyset$. Hence $(q - t) \in \overline{Appr}_{(K,r)}(T)$.
Let $q \in \overline{Appr}_{(K,r)}(T)$ and $a \in R$. Then $[q]_{(K,r)} \cap T \neq \emptyset$.
This implies there exist $z \in [q]_{(K,r)}$ and $z \in T$ such that $q \cdot r - z \cdot r \in K$.
Then $a \cdot (q \cdot r - z \cdot r) \in K$.
This gives $(a \cdot q) \cdot r - (a \cdot z) \cdot r \in K$.
This implies $(a \cdot z) \in [a \cdot q]_{(K,r)}$ and $(a \cdot z) \in T$ (As $T$ is a left ideal of $R$).
$\implies ([a \cdot q]_{(K,r)} \cap T) \neq \emptyset$. Hence $a \cdot q \in \overline{Appr}_{(K,r)}(T)$.
Thus $\overline{Appr}_{(K,r)}(T)$ is a left ideal of $R$. ■

**Corollary III.0.3.** If $K$ is an ideal of a commutative ring $R$ and $P$ is a subring of $R$, $r \in R$, then $\overline{Appr}_{(K,r)}(P)$ is a subring of $R$.

**Proposition III.19.** If $K$ is a right or left ideal of $R$ and $P$ is a subring of $R$, $r \in R$ and $\underline{Appr}_{(K,r)}(P) \neq \emptyset$, then $\underline{Appr}_{(K,r)}(P)$ is an additive subgroup of $R$.

*Proof:* Let $q, t \in \underline{Appr}_{(K,r)}(P)$. Then $[q]_{(K,r)} \subseteq P$ and $[t]_{(K,r)} \subseteq P$.

Now, we have to show that $[q-t]_{(K,r)} \subseteq P$. Let $z \in [q-t]_{(K,r)}$. Then, by Proposition III.6, we have $[q-t]_{(K,r)} = [q]_{(K,r)} - [t]_{(K,r)}$. Therefore $z \in [q]_{(K,r)} - [t]_{(K,r)}$. Then $z = z' - z''$ for some $z' \in [q]_{(K,r)}$ and $z'' \in [t]_{(K,r)}$. As $[q]_{(K,r)} \subseteq P$ and $[t]_{(K,r)} \subseteq P$, we get $z' \in P$ and $z'' \in P$.
This implies $(z' - z'') \in P$ (Since $P$ is a subring of $R$). This gives $z = z' - z'' \in P$.
Then $[q-t]_{(K,r)} \subseteq P$. Hence $(q-t) \in \underline{Appr}_{(K,r)}(P)$.
Thus $\underline{Appr}_{(K,r)}(P)$ is an additive subgroup of $R$. ■

**Proposition III.20.** If $K$ is a right or left ideal of $R$ and $P$ is a subring of $R$, $r \in R$, then $\overline{Appr}_{(K,r)}(P)$ is an additive subgroup of $R$.

**Proposition III.21.** If $K$ and $T$ are any two ideals of $R$ and $P$ is a subring of $R$, $r \in R$ and $\underline{Appr}_{(K+T,r)}(P) \neq \emptyset$, then $\underline{Appr}_{(K+T,r)}(P) \subseteq \underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P)$.

*Proof:* As $K \subseteq K+T$ and $T \subseteq K+T$, by Proposition III.15, we have $\underline{Appr}_{(K+T,r)}(P) \subseteq \underline{Appr}_{(K,r)}(P)$ and $\underline{Appr}_{(K+T,r)}(P) \subseteq \underline{Appr}_{(T,r)}(P)$.
This implies $\underline{Appr}_{(K+T,r)}(P) + \underline{Appr}_{(K+T,r)}(P) \subseteq \underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P)$. As $P$ is subring of $R$ and $\underline{Appr}_{(K+T,r)}(P) \neq \emptyset$, by Proposition III.19 we have $\underline{Appr}_{(K+T,r)}(P)$ is an additive subgroup of $R$. Then $\underline{Appr}_{(K+T,r)}(P) = \underline{Appr}_{(K+T,r)}(P) + \underline{Appr}_{(K+T,r)}(P)$.
Hence $\underline{Appr}_{(K+T,r)}(P) \subseteq \underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P)$. ■

**Proposition III.22.** If $K$ and $T$ are any two ideals of $R$ and $P$ is a subring of $R$, $r \in R$, then $\overline{Appr}_{(K,r)}(P) + \overline{Appr}_{(T,r)}(P) \subseteq \overline{Appr}_{(K+T,r)}(P)$.

*Proof:* As $K \subseteq K+T$ and $T \subseteq K+T$, by Proposition III.15. we have $\overline{Appr}_{(K,r)}(P) \subseteq \overline{Appr}_{(K+T,r)}(P)$ and $\overline{Appr}_{(T,r)}(P) \subseteq \overline{Appr}_{(K+T,r)}(P)$.
Then $\overline{Appr}_{(K,r)}(P) + \overline{Appr}_{(T,r)}(P) \subseteq \overline{Appr}_{(K+T,r)}(P) + \overline{Appr}_{(K+T,r)}(P)$.
By Proposition III.20, we have $\overline{Appr}_{(K+T,r)}(P)$ is an additive subgroup of $R$. Hence, we get $\overline{Appr}_{(K+T,r)}(P) + \overline{Appr}_{(K+T,r)}(P) = \overline{Appr}_{(K+T,r)}(P)$.
Thus $\overline{Appr}_{(K,r)}(P) + \overline{Appr}_{(T,r)}(P) \subseteq \overline{Appr}_{(K+T,r)}(P)$. ■

**Proposition III.23.** If $K$ and $T$ are ideals of a ring $R$ and $P$ is a subring of $R$, $r \in R$, then $\underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P) \subseteq \overline{Appr}_{(K+T,r)}(P)$.

*Proof:* Let $x \in \underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P)$. Then $x = u + v$ for some $u \in \underline{Appr}_{(K,r)}(P)$ and $v \in \underline{Appr}_{(T,r)}(P)$. This implies $[u]_{(K,r)} \subseteq P$ and $[v]_{(T,r)} \subseteq P$.
$\Longrightarrow [u]_{(K,r)} + [v]_{(T,r)} \subseteq P$(As $P$ is a subring of $R$).
By Proposition III.7, we have $[u]_{(K,r)} + [v]_{(T,r)} \subseteq [u+v]_{(K+T,r)}$.
$\Longrightarrow [u+v]_{(K+T,r)} \cap P \neq \emptyset \Longrightarrow x \in \overline{Appr}_{(K+T,r)}(P)$.
Hence $\underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P) \subseteq \overline{Appr}_{(K+T,r)}(P)$. ■

**Example III.9.** Let $R = \mathbb{Z}_6$ and $K = \{0, 3\}$ be an ideal of $R$. Then from the Example III.1, $K$ partitions $\mathbb{Z}_6$ into the equivalence classes as $[0]_{(K,2)} = \{0, 3\}$, $[1]_{(K,2)} = \{1, 4\}$ and $[2]_{(K,2)} = \{2, 5\}$.
Now, take $T = \{0, 2, 4\}$ is an ideal of $\mathbb{Z}_6$ and $P = \{0, 2, 4\}$ is a subring of $\mathbb{Z}_6$. Then $T = \{0, 2, 4\}$ partitions $\mathbb{Z}_6$ into the following equivalence classes.
$[0]_{(T,2)} = \{0, 2, 4\}$
$[1]_{(T,2)} = \{1, 3, 5\}$.
Here $K + T = \mathbb{Z}_6$. Then $\underline{Appr}_{(K,r)}(P) = \phi$ and $\underline{Appr}_{(T,r)}(P) = \{0, 2, 4\}$ and $\overline{Appr}_{(K+T,r)}(P) = \mathbb{Z}_6$.
$\underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P) = \{0, 2, 4\}$.
Hence $\underline{Appr}_{(K,r)}(P) + \underline{Appr}_{(T,r)}(P) \subset \overline{Appr}_{(K+T,r)}(P)$.

**Proposition III.24.** If $K$ is a left ideal of a ring $R$ and $r \in R$ such that $r^2 = r$, then $[x]_{([0]_{(K,r)},r)} \subseteq [x]_{(K,r)}$, $\forall\ x \in R$.

*Proof:* Let $y \in [x]_{([0]_{(K,r)},r)}$. Then $y \cdot r - x \cdot r \in [0]_{(K,r)}$. This implies $(y \cdot r - x \cdot r) \cdot r \in K$. Then $y \cdot r^2 - x \cdot r^2 \in K$. Hence, we get $y \cdot r - x \cdot r \in K$ (As $r^2 = r$).
Thus $[x]_{([0]_{(K,r)},r)} \subseteq [x]_{(K,r)}$. ■

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