Game-theoretical Evaluating Rules for Minimizing Losses under Multi-objective Situations

Hui-Chuan Wei, Ai-Tzu Li, Wei-Ni Wang and Yu-Hsien Liao

Abstract—**In the context of economic or managerial operational systems, the discussion often revolves around strategies to enhance efficiency. However, efficiency and loss are two sides of the same coin. Minimizing loss is crucial for the optimal functioning of operational systems, as reducing losses indirectly contributes to improving performance. Therefore, this study first proposes a rule within an operational system on how to evaluate and minimize losses under a multi-objective consideration. In real-world scenarios, participating entities and their levels of participation yield different effects across various objectives. Hence, corresponding weighted derivative rules are also introduced concerning participating entities, their participation levels, and related marginal effects. The mathematical correctness and practical applicability of these rules are also validated through related axiomatic processes.**

Index Terms—**Operational systems, multi-objective consideration, minimizing loss, rule, axiomatic process.**

I. Introduction

The enhancement of efficiency is a critical topic frequently addressed in economic or managerial operational systems. For instance, strategies on how to maximize commercial profits, increase production capacity in machines, or enhance the efficacy of pharmaceuticals are commonly explored. However, efficiency and loss are intricately linked. For instance, evaluating commercial profits requires considering production costs, assessing production capacity involves machine wear and tear, and evaluating pharmaceutical efficacy entails considering physiological risks. When assessing how to minimize losses, it often indirectly leads to enhancing efficiency. For example, reducing production costs ultimately increases net profits from product sales.

However, evaluating minimization of losses often necessitates considering multiple objectives. For instance, in selling products, reducing manufacturing costs conflicts with prolonging machine lifespan while simultaneously reducing pollution from the production process. These objectives sometimes clash, as extending machine lifespan and reducing pollution inevitably affect costs. In game theory, multi-objective analysis aims to derive equilibrium outcomes under various objectives. Under the field of mathematics, mathematical multi-objective game-theoretical methods are wildly applied to address such multi-objective consideration issues. The rules governing such situations lack acceptable structures to express reasonable consequences that, unlike usual perspectives or notions, evaluate numerous objective goals. Various existing researches have explored multi-objective situations. For instance, Bednarczuk et al. [1] transformed the multiple-choice knapsack problem into a bi-objective optimization problem, whose solution set encompasses solutions of the original multiple-choice knapsack problem. Goli et al. [3] addressed the optimization of the multivariate manufacturing portfolio problem under return uncertainty. The key achievement stems from employing an enhanced artificial intelligence-robust optimization hybrid approach, introducing a new concept for assessing the risk of a manufacturing portfolio. A bi-objective mathematical formulation (maximizing return and minimizing risk) is also presented. By delving into multi-attribute analysis techniques amidst diverse and complex conditions (e.g., considering multiple perspectives and incorporating multi-level participating entities), Guarini et al. [4] aim to outline a methodology for selecting the most suitable rule tailored to specific evaluation requirements, often encountered in strategic decision-making contexts. A resilient combinatorial optimization modeling approach by Mustakerov et al. [14] is advanced for multi-choice yield with diverse strategy maker prerequisites. This approach is founded on formulating multi-attribute linear mixed-integer optimization tasks. Tirkolaee et al. [18] highlighted the multi-attribute multi-mode utility-constrained manufacturing scheduling problem with compensation planning, where tasks can be completed through various modes, aiming to minimize completion time and maximize net present value simultaneously. Related studies also could be found in Cheng et al. [2], Khorram et al. [12], and so on.

Under traditional game-theoretical evaluating processes, participating entities typically consider participation or non-participation scenarios. The *equal allocation of non-separable costs* (EANSC, Ransmeier [16]) is a common efficiency evaluating rule proposed within traditional game theory. Moulin [13] defined the complement reduction and related consistency, combining other properties to prove that EANSC is a fair, just, and rational evaluating rule. To accommodate multi-objective considerations, participating entities in operational systems must have varying levels of involvement, necessitating a *multi-choice evaluating behavior* where each participating entity has different levels of participation. Under multi-choice situations, Hwang and Liao [7], Liao [8], [9], [11], and Nouweland et al. [15] have proposed several derivative evaluating rules and related axiomatic processes.

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In many operational systems, however, losses generated by participating entities or their levels of involvement often vary depending on the context. For example, the same pharmaceutical may cause different physiological harms in different treatment contexts, may interact differently with other drugs, and the dosage administered can also influence outcomes. Therefore, it is reasonable to assign corresponding *weights* based on the context for participating entities or their levels of involvement.

The above statements lead to a question:

- Can an evaluating rule be proposed based on multi-objectives, varying levels of participation, and relative weighting considerations on how to minimize losses under operational systems?

To address this question, the primary results presented in this study are as follows.

1) Inspired by evaluating notion of the EANSC, within a multi-objective multi-choice structure, we propose a rule for evaluating and minimizing loss under operational systems, termed as the *minimal uniform evaluation of indivisible losses* (MUEIL).
2) Since losses generated by participating entities or their levels of involvement often vary across different contexts, we define different weight functions for participating entities and their levels of involvement. These two weight functions are integrated into the concept of the MUEIL, resulting in two weighted derivative forms.
3) However, defining weight functions and assigning weights to participating entities or their levels of involvement may be somewhat artificial. Therefore, this study utilizes marginal effects from participation as substitutes for weights, deriving a natural weighted form as a result.
4) Further, we extend the Moulin's [13] reduction, its corresponding consistency and different balance properties to the structure of multi-objective multi-choice situations, and utilize related axiomatic processes to demonstrate the mathematical correctness and practical applicability of the evaluating rules proposed in this study.

II. Preliminaries

A. Definitions and notations

Let $\mathbb{UPE}$ denote the universe of participating entities, for instance, the set comprised of all citizens of a country. Any $b \in \mathbb{UPE}$ is identified as a participating entity of $\mathbb{UPE}$, such as a citizen in a country. For $b \in \mathbb{UPE}$ and $\xi_b \in \mathbb{N}$, we define $\mathbb{L}_b = \{0, 1, \cdots, \xi_b\}$ to represent the set of participation levels for participating entity b, and $\mathbb{L}_b^+ = \mathbb{L}_b \setminus \{0\}$, where 0 indicates no operation.

Consider $\mathbb{P} \subseteq \mathbb{UPE}$ as the largest set encompassing all participating entities of an interactive system within $\mathbb{UPE}$, like all employees of a company in a country. Let $\mathbb{L}^{\mathbb{P}} = \prod_{b \in \mathbb{P}} \mathbb{L}_b$ be the product set of participation level sets for every participating entity in $\mathbb{P}$. For every $Q \subseteq \mathbb{P}$, a participating entity alliance Q corresponds, in a standard manner, to the multi-choice alliance $\hat{z}^Q \in \mathbb{L}^{\mathbb{P}}$, which is a vector indicating $\hat{z}^Q_q = 1$ if $q \in Q$, and $\hat{z}^Q_q = 0$ if $q \in \mathbb{P} \setminus Q$. Denote $0_{\mathbb{P}}$ as the zero vector in $\Re^{\mathbb{P}}$. For $m \in \mathbb{N}$, also define 0_m as the zero vector in $\Re^m$ and $\mathbb{M}_m = \{1, 2, \cdots, m\}$.

A **multi-choice circumstance** is denoted as $(\mathbb{P}, \xi, \chi)$, where $\mathbb{P} \neq \emptyset$ is a finite set of participating entities, $\xi = (\xi_b)_{b \in \mathbb{P}} \in \mathbb{L}^{\mathbb{P}}$ is a vector indicating the number of participation levels for each participating entity, and $\chi : \mathbb{L}^{\mathbb{P}} \to \Re$ is a mapping with $\chi(0_{\mathbb{P}}) = 0$ that assigns to each participation level vector $\mu = (\mu_b)_{b \in \mathbb{P}} \in \mathbb{L}^{\mathbb{P}}$ the benefit that participating entities can receive when operating at level μ_b. A **multi-objective multi-choice circumstance** is denoted by $(\mathbb{P}, \xi, \mathbb{X}^m)$, where $m \in \mathbb{N}$, $\mathbb{X}^m = (\chi^t)_{t \in \mathbb{M}_m}$ and $(\mathbb{P}, \xi, \chi^t)$ represents a multi-choice circumstance for each $t \in \mathbb{M}_m$. The family of all multi-objective multi-choice circumstances is denoted as $\mathbb{MMC}$.

A **rule** is defined as a mapping ρ that assigns to each $(\mathbb{P}, \xi, \mathbb{X}^m) \in \mathbb{MMC}$ an element

$$\rho\big(\mathbb{P}, \xi, \mathbb{X}^m\big) = \big(\rho^t\big(\mathbb{P}, \xi, \mathbb{X}^m\big)\big)_{t \in \mathbb{M}_m},$$

where $\rho^t\big(\mathbb{P}, \xi, \mathbb{X}^m\big) = \big(\rho^t_b\big(\mathbb{P}, \xi, \mathbb{X}^m\big)\big)_{b \in \mathbb{P}} \in \Re^{\mathbb{P}}$ and $\rho^t_b\big(\mathbb{P}, \xi, \mathbb{X}^m\big)$ represents the remuneration of participating entity b when b operates in $\big(\mathbb{P}, \xi, \chi^t\big)$. Let $(\mathbb{P}, \xi, \mathbb{X}^m) \in \mathbb{MMC}$, $Q \subseteq \mathbb{P}$, and $\mu \in \Re^{\mathbb{P}}$. We define $N(\mu) = \{b \in \mathbb{P} | \mu_b \neq 0\}$ and $\mu_Q \in \Re^Q$ as the restriction of μ to Q. Given $b \in \mathbb{P}$, we also define μ_{-b} to represent $\mu_{\mathbb{P} \setminus \{b\}}$. Additionally, $\sigma = (\mu_{-b}, i) \in \Re^{\mathbb{P}}$ is defined by $\sigma_{-b} = \mu_{-b}$ and $\sigma_b = i$.

Based on the claim of evaluating how to minimize various losses during operational processes, this study introduces derivative the concepts of the EANSC within the framework of multi-objective multi-choice circumstances.

Definition 1: The **minimal uniform evaluation of indivisible losses (MUEIL)**, $\overline{\Delta}$, is defined by

$$\overline{\Delta^t_b}(\mathbb{P}, \xi, \mathbb{X}^m) = \Delta^t_b(\mathbb{P}, \xi, \mathbb{X}^m) + \frac{1}{|\mathbb{P}|}\big[\chi^t(\xi) - \sum_{k \in \mathbb{P}} \Delta^t_k(\mathbb{P}, \xi, \mathbb{X}^m)\big]$$

for every $(\mathbb{P}, \xi, \mathbb{X}^m) \in \mathbb{MMC}$, for every $t \in \mathbb{M}_m$ and for every $b \in \mathbb{P}$. The value

$$\Delta^t_b(\mathbb{P}, \xi, \mathbb{X}^m) = \min_{q \in \mathbb{L}^+_b} \{\chi^t(\xi_{-b}, q) - \chi^t(\xi_{-b}, q-1)\}$$

is the **minimal marginal level-loss** among all participation levels of participating entity b in $(\mathbb{P}, \xi, \chi^t)$.[1] Under the notion of $\overline{\Delta}$, all participating entities firstly evaluate its minimal marginal level-losses, and further assess the rest of losses equally.

As mentioned in the introduction, related concept of weights often becomes important consideration under various allocating processes. For example, allocating proportions for weights may be related to asset allocation, where weights can represent the relative investment risk of various asset plans under the investment circumstance. Similarly, weights can be applied in insurance contracts, where the risk of various insurance items relative to different employment circumstances may result in different weighted premium costs. Even if the insurance items and employment circumstances for a particular insurance product are fixed, the premium costs of insurance items may vary in weighted proportions relative to different insured parties. Therefore, it is quite reasonable to

[1]This study utilizes bounded multi-choice circumstances, treated as the circumstances $(\mathbb{P}, \xi, \chi^t)$ such that, there exists $B^t_\chi \in \Re$ such that $\chi^t(\mu) \leq B^t_\chi$ for every $\mu \in \mathbb{L}^{\mathbb{P}}$. It could be utilized to assure that $\Delta^t_b(\mathbb{P}, \xi, \chi^t)$ is well-defined.

assign weights to "participating entities" or their "participation levels" to differentiate their relative differences.

Let $\Omega : \mathbb{UPE} \to \mathbb{R}^+$ be a positive map. Then Ω is treated as a **weight function for participating entities**. Similarly, let $\Xi : \cup_{b\in\mathbb{UPE}}\mathbb{L}_b^+ \to \mathbb{R}^+$ be a positive map. Then Ξ is treated as a **weight function for participation levels**. Based on these two kinds of weight functions, three weighted extensions of the MUEIL could be considered as follows.

Definition 2:

- The **1-minimal weighted evaluation of indivisible losses (1-MWEIL)**, Δ^Ω, is defined as follows: For every $(\mathbb{P}, \xi, \mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participating entities Ω, for every $t \in \mathbb{M}_m$, and for every participating entity $b \in \mathbb{P}$,

$$\begin{aligned}&\Delta_b^{\Omega,t}(\mathbb{P}, \xi, \mathbb{X}^m)\\ =\;& \Delta_b^t(\mathbb{P}, \xi, \mathbb{X}^m) + \frac{\Omega(b)}{\sum\limits_{k\in\mathbb{P}}\Omega(k)}\left[\chi^t(\xi) - \sum_{k\in\mathbb{P}}\Delta_k^t(\mathbb{P}, \xi, \mathbb{X}^m)\right].\end{aligned}$$

  According to the definition of Δ^Ω, all participating entities initially evaluate their minimal marginal level-losses, and the remaining losses are assessed proportionally via weights for participating entities.
- The **2-minimal weighted evaluation of indivisible losses (2-MWEIL)**, Δ^Ξ, is defined as follows: For every $(\mathbb{P}, \xi, \mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participation levels Ξ, for every $t \in \mathbb{M}_m$, and for every participating entity $b \in \mathbb{P}$,

$$\begin{aligned}&\Delta_b^{\Xi,t}(\mathbb{P}, \xi, \mathbb{X}^m)\\ =\;& \Gamma_b^{\Xi,t}(\mathbb{P}, \xi, \mathbb{X}^m) + \frac{1}{|\mathbb{P}|}\left[\chi^t(\xi) - \sum_{k\in\mathbb{P}}\Gamma_k^{\Xi,t}(\mathbb{P}, \xi, \mathbb{X}^m)\right],\end{aligned}$$

  where

$$\Gamma_b^{\Xi,t}(\mathbb{P}, \xi, \mathbb{X}^m) = \min_{q\in\mathbb{L}_b^+}\{\Xi(q)\cdot[\chi^t(\xi_{-b}, q) - \chi^t(\xi_{-b}, q-1)]\}$$

  is the minimal weighted marginal level-loss among all participation levels of participating entity b. By definition of $\Delta^{\Xi,t}$, all participating entities initially evaluate their minimal weighted marginal level-losses, and the remaining losses are assessed equally.
- The **bi-weighted evaluation of indivisible losses (BWEIL)**, $\Delta^{\Omega,\Xi}$, is defined by for every $(\mathbb{P}, \xi, \mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t \in \mathbb{M}_m$ and for every participating entity $b \in \mathbb{P}$,

$$\begin{aligned}&\Delta_b^{\Omega,\Xi,t}(\mathbb{P}, \xi, \mathbb{X}^m)\\ =\;& \Gamma_b^{\Xi,t}(\mathbb{P}, \xi, \mathbb{X}^m) + \frac{\Omega(b)}{\sum\limits_{k\in\mathbb{P}}\Omega(k)}\left[\chi^t(\xi) - \sum_{k\in\mathbb{P}}\Gamma_k^{\Xi,t}(\mathbb{P}, \xi, \mathbb{X}^m)\right].\end{aligned}$$

  Based on the definition of $\Delta^{\Omega,\Xi}$, all participating entities initially evaluate their minimal weighted marginal level-losses, and the remaining losses are assessed proportionally via weights for participating entities.

B. Motivating and practical examples

As mentioned in the introduction, each participating entity can choose different participation levels relative to different real-world situations. Moreover, multi-objective game-theoretical analysis considers how the operational processes and outcomes of systems operate in equilibrium when multiple criteria are simultaneously involved. Related concepts have been applied in many fields, including industrial engineering, ecological assessments, drug trials, logistics management, and decision-making sciences, where balancing and formulating equilibrium strategies are needed under considerations of multiple objectives. For example, in the plant assessment process of a technology company, how to reduce costs while delaying machine wear and tear, reducing pollution emissions, and resource consumption, but still maintaining or even improving efficiency. In many cases, three or more objectives may be involved. Therefore, this study focuses on the framework of multi-objective multi-choice considerations.

However, it may not always be appropriate to consider the importance of participating entities or their participation levels as equal under different circumstances. Therefore, it is very reasonable to assign relative weights to participating entities or their participation levels under operating processes and distributing processes relative to different operational circumstances. Thus, the advantage of the evaluating rules proposed in this study lie in considering multiple objectives, multiple participation levels, and relative weighting, confirming that the global value of each participating entity can be derived through evaluating minimization of losses.

To illustrate the concept of multi-objective multi-choice scenarios, let us consider a concise example in the context of "risk management." Let $\mathbb{P}$ represent the set formed by all participating entities in the operational organization $(\mathbb{P}, \xi, \mathbb{X}^m)$. The function χ^t can be seen as a risk evaluating, evaluating the risk value of each overall participation level vector $\alpha = (\alpha_b)_{b\in\mathbb{P}} \in \mathbb{L}^{\mathbb{P}}$, which presents the risk generated by each entity $b \in \mathbb{P}$ when operating with a specific participation level $\alpha_b \in \mathbb{L}_b$ in the organizational sub-department $(\mathbb{P}, \xi, \chi^t)$. Using this corresponding framework, an operational organization $(\mathbb{P}, \xi, \mathbb{X}^m)$ can be expanded to a multi-objective multi-choice circumstance, where χ^t represents each risk evaluating function, and $\mathbb{L}_b$ represents the set formed by all participation levels of each participating entity b. Subsequently, we will further propose how to apply the rules defined in this study for minimizing losses in practical applications.

In the operational system example mentioned above, all participating entities not only incur their unique losses within the overall operational circumstance but may also experience different effects due to interactions with other participating entities. These effects can lead to mutual resistance, exacerbated losses, or unpredictable damages. As demonstrated in the example above, the mapping χ^t can be seen as an evaluation of loss effects, where the collaborative participation levels of all participating entities can be represented by the vector $\alpha = (\alpha_b)_{b\in\mathbb{P}} \in \mathbb{L}^{\mathbb{P}}$.

- According to the evaluating notion of the MUEIL defined in Definition 1, one could first evaluate the minimal marginal level-losses caused by the participation levels of each participating entity, and the remaining losses will be shared by all participants.
- However, since each type of participating entity may incur different relative loss effects relative to different operational departments or related businesses, it is reasonable to generate weights via weight function for participating entities Ω. Initially evaluating the minimal marginal level-losses caused by the participation levels of each participating entity, the remaining losses should be allocated based on the relative weighted proportions of each participating entity, which is the evaluating notion of the 1-MWEIL in Definition 2.

- On the other hand, since the participation levels of each participating entity may result in different loss effects relative to different operational departments or related businesses, these participation levels naturally exhibit varying degrees of relativity. Therefore, it is reasonable to generate weights through weight function for participation levels Ξ. Initially evaluating the minimal weighted marginal level-losses caused by the participation levels of each participating entity, the remaining losses will be shared by all participating entities, which is the evaluating notion of the 2-MWEIL in Definition 2.
- If we combine the concepts of the 1-MWEIL and the 2-MWEIL, initially evaluating the minimal weighted marginal level-losses caused by the participation levels of each participating entity, the remaining loss effects should be allocated based on the relative weighted proportions of each participating entity, which is the evaluating notion of the BWEIL in Definition 2.

III. Axiomatic processes

A. Axiomatic results for the MUEIL and its weighted extensions

Inspired by the axiomatic concepts due to Hart and Mas-Colell [5] and Moulin [13], several axiomatic results for the MUEIL, the 1-MWEIL, the 2-MWEIL, and the BWEIL are presented below to demonstrate mathematical correctness and the practicality of these rules.

Definition 3: A rule ρ fits the **multi-objective effectiveness (MOEIS)** requirement if, for every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$ and for every $t\in\mathbb{M}_m$, the sum of remunerations evaluated via ρ to all participating entities in $\mathbb{P}$ coincides with the overall effect $\chi^t(\xi)$. The MOEIS requirement ensures that all participating entities evaluate whole the losses entirely.

Lemma 1: The rules $\overline{\Delta}$, Δ^{Ω}, Δ^{Ξ}, $\Delta^{\Omega,\Xi}$ fit MOEIS.

Proof of Lemma 1: Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, $t\in\mathbb{M}_m$, Ω be weight function for participating entities and Ξ be weight function for participation levels. By Definition 2,

$$\begin{aligned}
&\sum_{b\in\mathbb{P}}\Delta_b^{\Omega,\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\\
=\;&\sum_{b\in\mathbb{P}}\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\\
&+\sum_{b\in\mathbb{P}}\Big[\frac{\Omega(b)}{\sum_{k\in\mathbb{P}}\Omega(k)}\big[\chi^t(\xi)-\sum_{k\in\mathbb{P}}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\big]\Big]\\
=\;&\sum_{b\in\mathbb{P}}\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)+\frac{\sum_{b\in\mathbb{P}}\Omega(b)}{\sum_{k\in\mathbb{P}}\Omega(k)}\big[\chi^t(\xi)-\sum_{k\in\mathbb{P}}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\big]\\
=\;&\sum_{b\in\mathbb{P}}\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)+\chi^t(\xi)-\sum_{k\in\mathbb{P}}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\\
=\;&\chi^t(\xi).
\end{aligned}$$

The proof is done. If all the weights for participating entities are set to 1 in the above proof process, the MOEIS requirement of 2-MWEIL can be verified. Similarly, if all the weights for participation levels are set to 1 in the above proof process, the MOEIS requirement of 1-MWEIL can be completed. Furthermore, if all the weights for both participating entities and participation levels are set to 1 in the above proof process, the MOEIS requirement of MUEIL can be finished. ■

To characterize the EANSC, Moulin [13] proposed the concept of the complement reduction, where each coalition within a subgroup can only receive its payoff if the coalition's payoff matches the original payoff of all non-participating entities outside the subgroup. The derived definition of complement reduction under multi-objective multi-choice circumstances is introduced as follows.

Definition 4:

- Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, $K\subseteq\mathbb{P}$, and ρ be a rule. The **reduced circumstance** $(K,\xi_K,\mathbb{X}^m_{K,\rho})$ is defined by $\mathbb{X}^m_{K,\rho}=(\chi^t_{K,\rho})_{t\in\mathbb{M}_m}$, and for every $\mu\in\mathbb{L}^K$,

$$\chi^t_{K,\rho}(\mu)=\begin{cases}0 & \text{if } \mu=0_K,\\ \chi^t(\mu,\xi_{\mathbb{P}\setminus K})-\sum_{b\in\mathbb{P}\setminus K}\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) & \text{otherwise,}\end{cases}$$

- Moreover, a rule ρ satisfies the **multi-objective bilateral consistency (MOBCY)** requirement if $\rho^t_b(K,\xi_K,\mathbb{X}^m_{K,\rho})=\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m)$ for every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, for every $t\in\mathbb{M}_m$, for every $K\subseteq\mathbb{P}$ with $|K|=2$, and for every $b\in K$.

Lemma 2: The rules $\overline{\Delta}$, Δ^{Ω}, Δ^{Ξ}, $\Delta^{\Omega,\Xi}$ fit MOBCY.

Proof of Lemma 2: Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, $K\subseteq\mathbb{P}$, $t\in\mathbb{M}_m$, Ω be weight function for participating entities and Ξ be weight function for participation levels. Let $|\mathbb{P}|\geq 2$ and $|K|=2$. By Definition 2,

$$\begin{aligned}
&\Delta_b^{\Omega,\Xi,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi}})\\
=\;&\Gamma_b^{\Xi,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi}})\\
&+\frac{\Omega(b)}{\sum_{k\in K}\Omega(k)}\cdot\big[\chi^t_{K,\Delta^{\Omega,\Xi}}(\xi_K)-\sum_{k\in K}\Gamma_k^{\Xi,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi}})\big]
\end{aligned}\tag{1}$$

for every $b\in K$ and for every $t\in\mathbb{M}_m$. By definitions of $\Gamma^{\Xi,t}$ and $\chi^t_{K,\Delta^{\Omega,\Xi}}$,

$$\begin{aligned}
&\Gamma_b^{\Xi,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi}})\\
=\;&\min_{q\in\mathbb{L}_b^+}\{\Xi(q)[\chi^t_{K,\Delta^{\Omega,\Xi}}(\xi_{K\setminus\{b\}},q)-\chi^t_{K,\Delta^{\Omega,\Xi}}(\xi_{K\setminus\{b\}},q-1)]\}\\
=\;&\min_{q\in\mathbb{L}_b^+}\{\Xi(q)[\chi^t(\xi_{-b},q)-\chi^t(\xi_{-b},q-1)]\}\\
=\;&\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m).
\end{aligned}\tag{2}$$

Based on equations (1), (2) and definitions of $\chi^t_{K,\Delta^{\Omega,\Xi}}$ and $\Delta^{\Omega,\Xi}$,

$$\begin{aligned}
&\Delta_b^{\Omega,\Xi,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi}})\\
=\;&\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\\
&+\frac{\Omega(b)}{\sum_{k\in K}\Omega(k)}\big[\chi^t_{K,\Delta^{\Omega,\Xi}}(\xi_K)-\sum_{k\in K}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\big]\\
=\;&\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)+\frac{\Omega(b)}{\sum_{k\in K}\Omega(k)}\big[\chi^t(\xi)-\sum_{k\in\mathbb{P}\setminus K}\Delta_k^{\Omega,\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\\
&-\sum_{k\in K}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\big]\\
=\;&\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\\
&+\frac{\Omega(b)}{\sum_{k\in K}\Omega(k)}\big[\sum_{k\in K}\Delta_k^{\Omega,\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\sum_{k\in K}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\big]\\
&\textbf{(MOEIS of } \Delta^{\Omega,\Xi}\textbf{)}\\
=\;&\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\\
&+\frac{\Omega(b)}{\sum_{k\in K}\Omega(k)}\Big[\frac{\sum_{k\in K}\Omega(k)}{\sum_{p\in\mathbb{P}}\Omega(p)}\big[\chi^t(\xi)-\sum_{p\in\mathbb{P}}\Gamma_{\mathbb{P}}^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\big]\Big]\\
=\;&\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)+\frac{\Omega(b)}{\sum_{p\in\mathbb{P}}\Omega(p)}\big[\chi^t(\xi)-\sum_{p\in\mathbb{P}}\Gamma_{\mathbb{P}}^{t}(\mathbb{P},\xi,\mathbb{X}^m)\big]\\
=\;&\Delta_b^{\Omega,\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)
\end{aligned}$$

for every $b\in K$ and for every $t\in\mathbb{M}_m$. If all the weights for participating entities are set to 1 in the above proof process, the MOBCY requirement of 2-MWEIL can be verified. Similarly, if all the weights for participation levels are set to 1 in the above proof process, the MOBCY requirement of 1-MWEIL can be completed. Furthermore, if all the weights for both participating entities and participation levels are set to 1 in the above proof process, the MOBCY requirement of MUEIL can be finished. ■

Definition 5:

- A rule ρ satisfies the **multi-objective rule for circumstances (MORC)** requirement if

$\rho(\mathbb{P},\xi,\mathbb{X}^m) = \overline{\Delta}(\mathbb{P},\xi,\mathbb{X}^m)$ for every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$ with $|\mathbb{P}| \leq 2$.

- rm A rule ρ satisfies the **1-weighted rule for circumstances (1WRFC)** if $\rho(\mathbb{P},\xi,\mathbb{X}^m) = \Delta^{\Omega}(\mathbb{P},\xi,\mathbb{X}^m)$ for every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$ with $|\mathbb{P}| \leq 2$ and for every weight function Ω for participating entities.
- A rule ρ satisfies the **2-weighted rule for circumstances (2WRFC)** if $\rho(\mathbb{P},\xi,\mathbb{X}^m) = \Delta^{\Xi}(\mathbb{P},\xi,\mathbb{X}^m)$ for every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$ with $|\mathbb{P}| \leq 2$ and for every weight function Ξ for levels.
- A rule ρ fits **bi-weighted rule for circumstances (BWRC)** if $\rho(\mathbb{P},\xi,\mathbb{X}^m) = \Delta^{\Omega,\Xi}(\mathbb{P},\xi,\mathbb{X}^m)$ for every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$ with $|\mathbb{P}| \leq 2$, for every weight function for participating entities Ω and for every weight function for participation levels Ξ.
- The requirements of MORC, MORC, 1WRFC, 2WRFC, BWRC are generalized analogues of the two-entry rule standard proposed by Hart and Mas-Colell [5].

Inspired by the axiomatic concepts due to Hart and Mas-Colell [5] and Moulin [13], the MOBCY requirement would be adopted to characterize these evaluating rules.

Theorem 1:

1) On $\mathbb{MMC}$, the MUEIL is the unique rule fitting MORC and MOBCY.
2) On $\mathbb{MMC}$, the 1-MWEIL is the unique rule fitting 1WRFC and MOBCY.
3) On $\mathbb{MMC}$, the 2-MWEIL is the unique rule fitting 2WRFC and MOBCY.
4) On $\mathbb{MMC}$, the BWEIL is the unique rule fitting BWRC and MOBCY.

Proof of Theorem 2: By Lemma 2, the rules $\overline{\Delta}$, Δ^{Ω}, Δ^{Ξ}, $\Delta^{\Omega,\Xi}$ fit MOBCY. Clearly, the rules $\overline{\Delta}$, Δ^{Ω}, Δ^{Ξ}, $\Delta^{\Omega,\Xi}$ fit MORC, 1WRFC, 2WRFC and BWRC respectively.

To present the uniqueness of result 4, suppose that ρ fits BWRC and MOBCY. By BWRC and MOBCY of ρ, it is easy to clarify that ρ also fits MOEIS, thus we omit it. Let $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, Ω be weight function for participating entities and Ξ be weight function for participation levels. By BWRC of ρ, $\rho(\mathbb{P},\xi,\mathbb{X}^m) = \Delta^{\Omega,\Xi}(\mathbb{P},\xi,\mathbb{X}^m)$ if $|\mathbb{P}| \leq 2$. The situation $|\mathbb{P}| > 2$: Let $b \in \mathbb{P}$, $t \in \mathbb{M}_m$ and $K = \{b,p\}$ with $p \in \mathbb{P} \setminus \{b\}$.

$$\begin{aligned} & \rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) - \Delta^{\Omega,\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m) \\ = \; & \rho^t_b(K,\xi_K,\mathbb{X}^m_{K,\rho}) - \Delta^{\Omega,\Xi,t}_b(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi}}) \\ & \textbf{(MOBCY of } \Delta^{\Omega,\Xi,t} \textbf{ and } \rho\textbf{)} \\ = \; & \Delta^{\Omega,\Xi,t}_b(K,\xi_K,\mathbb{X}^m_{K,\rho}) - \Delta^{\Omega,\Xi,t}_b(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi,}}). \\ & \textbf{(BWRC of } \rho\textbf{)} \end{aligned} \tag{3}$$

Similar to equation (2)

$$\Gamma^{\Xi,t}_b(K,\xi_K,\mathbb{X}^m_{K,\rho}) = \Gamma^{\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m) = \Gamma^{\Xi,t}_b(K,\xi_K,\mathbb{X}^m_{K,\Gamma^{\Delta,\Xi,}}). \tag{4}$$

By equations (3) and (4),

$$\begin{aligned} & \rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) - \Delta^{\Omega,\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m) \\ = \; & \Delta^{\Omega,\Xi,t}_b(K,\xi_K,\mathbb{X}^m_{K,\rho}) - \Delta^{\Omega,\Xi,t}_b(K,\xi_K,\mathbb{X}^m_{K,\Delta^{\Omega,\Xi,}}) \\ = \; & \tfrac{\Omega(b)}{\Omega(b)+\Omega(p)} \cdot \left[\chi^t_{K,\rho}(\xi_K) - \chi^t_{K,\Delta^{\Omega,\Xi}}(\xi_K)\right] \\ = \; & \tfrac{\Omega(b)}{\Omega(b)+\Omega(p)} \cdot \big[\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) + \rho^t_p(\mathbb{P},\xi,\mathbb{X}^m) \\ & \quad -\Delta^{\Omega,\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m) - \Delta^{\Omega,\Xi,t}_p(\mathbb{P},\xi,\mathbb{X}^m)\big]. \end{aligned}$$

Thus,

$$\begin{aligned} & \Omega(p) \cdot \left[\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) - \Delta^{\Omega,\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = \; & \Omega(b) \cdot \left[\rho^t_p(\mathbb{P},\xi,\mathbb{X}^m) - \Delta^{\Omega,\Xi,t}_p(\mathbb{P},\xi,\mathbb{X}^m)\right]. \end{aligned}$$

By MOEIS of $\Delta^{\Omega,\Xi,t}$ and ρ,

$$\begin{aligned} & \left[\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) - \Delta^{\Omega,\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m)\right] \cdot \sum_{p\in\mathbb{P}} \Omega(p) \\ = \; & \Omega(b) \cdot \sum_{p\in\mathbb{P}} \left[\rho^t_p(\mathbb{P},\xi,\mathbb{X}^m) - \Delta^{\Omega,\Xi,t}_p(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = \; & \Omega(b) \cdot \left[\chi^t(\xi) - \chi^t(\xi)\right] \\ = \; & 0. \end{aligned}$$

Hence, $\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) = \Delta^{\Omega,\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m)$ for every $b \in \mathbb{P}$ and for every $t \in \mathbb{M}_m$. If all the weights for participating entities are set to 1 in the above proof process, the proof of outcome 3 could be verified. Similarly, if all the weights for participation levels are set to 1 in the above proof process, the proof of outcome 2 could be completed. Furthermore, if all the weights for both participating entities and participation levels are set to 1 in the above proof process, the proof of outcome 1 could be presented. ■

In the following some instances are exhibited to display that every of the requirements applied in Theorem 1 is independent of the rest of requirements.

Example 1: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t \in \mathbb{M}_m$ and for every participating entity $b \in \mathbb{P}$,

$$\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) = \begin{cases} \Delta^{\Omega,\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m) & \text{if } |\mathbb{P}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ρ fits BWRC, but it does not fit MOBCY.

Example 2: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t \in \mathbb{M}_m$ and for every participating entity $b \in \mathbb{P}$,

$$\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) = \begin{cases} \Delta^{\Xi,t}_b(\mathbb{P},\xi,\mathbb{X}^m) & \text{if } |\mathbb{P}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ρ fits 2WRFC, but it does not fit MOBCY.

Example 3: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t \in \mathbb{M}_m$ and for every participating entity $b \in \mathbb{P}$,

$$\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) = \begin{cases} \Delta^{\Omega,t}_b(\mathbb{P},\xi,\mathbb{X}^m) & \text{if } |\mathbb{P}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ρ fits 1WRFC, but it does not fit MOBCY.

Example 4: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t \in \mathbb{M}_m$ and for every participating entity $b \in \mathbb{P}$,

$$\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) = \begin{cases} \overline{\Delta^t_b}(\mathbb{P},\xi,\mathbb{X}^m) & \text{if } |\mathbb{P}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ρ fits MORC, but it does not fit MOBCY. Δ^{Ω}

Example 5: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t \in \mathbb{M}_m$ and for every participating entity $b \in \mathbb{P}$, $\rho^t_b(\mathbb{P},\xi,\mathbb{X}^m) = 0$. Clearly, ρ fits MOBCY, but it does not fit MORC, 1WRFC, 2WRFC and BWRC.

B. Different generalization and revised consistency

In Section 2 and Section 3.1, this study respectively proposes weighted rules for evaluating the corresponding weighted allocating notions concerning participating entities and their participation levels. However, the fairness or legitimacy of these weighted functions may be questioned, meaning the relative weighting of participating entities or their participation levels may seem somewhat arbitrary. Therefore, replacing weighted functions with relative minimal marginal level-losses in different situations appears to be more reasonable.

By substituting "minimal marginal level-losses" for "weights", the following loss evaluating notion can be defined differently from previous ones.

Definition 6: The **multi-choice multi-objective interior rule (MMOIR)**, Δ^I, is defined as follows: for every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}^*$, for every $t \in \mathbb{M}_m$, and for every participating entity $b \in \mathbb{P}$,

$$\begin{aligned} & \Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = \ & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) + \frac{\Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{k\in\mathbb{P}} \Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m)} \cdot \big[\chi^t(\xi) - \sum_{k\in\mathbb{P}} \Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m)\big], \end{aligned}$$

where

$$\mathbb{MMC}^* = \{(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC} | \sum_{k\in\mathbb{P}} \Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m) \neq 0 \ \forall\, t \in \mathbb{M}_m\}.$$

Based on definition of Δ^I, all participating entities initially evaluate their minimal marginal level-losses, and the remaining losses then assessed proportionally based on these minimal marginal level-losses.

Next, one would like to axiomatize the MMOIR using related notion of consistency. It is straightforward to verify that $\sum_{k\in K} \Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m) = 0$ for some $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for some $K \subseteq \mathbb{P}$, and for some $t \in \mathbb{M}_m$, i.e., $\Delta^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta})$ doesn't exist for some $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for some $K \subseteq \mathbb{P}$, and for some $t \in \mathbb{M}_m$. Therefore, we focus on the *multi-objective revised consistency* as follows.

Definition 7:

- A rule ρ fits the **multi-objective revised-consistency (MORCON)** if $(K,\xi_K,\mathbb{X}^m_{K,\rho})$ and $\rho(K,\xi_K,\mathbb{X}^m_{K,\rho})$ exist for some $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, for some $K \subseteq \mathbb{P}$, and for some $t \in \mathbb{M}_m$, and it holds that $\rho_b(K,\xi_K,\mathbb{X}^m_{K,\rho}) = \rho_b(\mathbb{P},\xi,\mathbb{X}^m)$ for every $b \in K$.
- A rule ρ fits the **multi-objective interior standard (MOIS)** if $\rho(\mathbb{P},\xi,\mathbb{X}^m) = \Delta^I(\mathbb{P},\xi,\mathbb{X}^m)$ for every $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$ with $|\mathbb{P}| \leq 2$.

Similar to Theorem 1, the related axiomatic process of Δ^I can also be presented as follows.

Lemma 3: The rule Δ^I fits MOEIS on $\mathbb{MMC}^*$.

Proof of Lemma 3: Let $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}$, $t \in \mathbb{M}_m$, Ω be weight function for participating entities and Ξ be weight function for participation levels. By Definition 6,

$$\begin{aligned} & \sum_{b\in\mathbb{P}} \Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = \ & \sum_{b\in\mathbb{P}} \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) \\ & + \sum_{b\in\mathbb{P}} \Big[\frac{\Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{k\in\mathbb{P}} \Delta_K^t(\mathbb{P},\xi,\mathbb{X}^m)} \big[\chi^t(\xi) - \sum_{k\in\mathbb{P}} \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m)\big]\Big] \\ = \ & \sum_{b\in\mathbb{P}} \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) + \chi^t(\xi) - \sum_{k\in\mathbb{P}} \Delta_K^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = \ & \chi^t(\xi). \end{aligned}$$

The proof is done. ∎

Lemma 4: The rule Δ^I fits MORCON on $\mathbb{MMC}^*$.

Proof of Lemma 4: Let $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}^*$, $K \subseteq \mathbb{P}$, $t \in \mathbb{M}_m$, Ω be weight function for participating entities and Ξ be weight function for participation levels. Let $|\mathbb{P}| \geq 2$ and $|K| = 2$. By Definition 6,

$$\begin{aligned} & \Delta_b^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^I}) \\ = \ & \Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\Delta^I}) \\ & + \frac{\Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\Delta^I})}{\sum_{k\in K} \Delta_K^t(K,\xi_K,\mathbb{X}^m_{K,\Delta^I})} \big[\chi^t_{K,\Delta^I}(\xi_K) \\ & \qquad - \sum_{k\in K} \Delta_k^t(K,\xi_K,\mathbb{X}^m_{K,\Delta^I})\big] \end{aligned} \tag{5}$$

for every $b \in K$ and for every $t \in \mathbb{M}_m$. By definitions of Δ^I and χ^t_{K,Δ^I},

$$\begin{aligned} & \Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\Delta^I}) \\ = \ & \min_{q\in\mathbb{L}_b^+} \{\chi^t_{K,\Delta^I}(\xi_{K\setminus\{b\}},q) - \chi^t_{K,\Delta^I}(\xi_{K\setminus\{b\}},q-1)\} \\ = \ & \min_{q\in\mathbb{L}_b^+} \{\chi^t(\xi_{-b},q) - \chi^t(\xi_{-b},q-1)\} \\ = \ & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m). \end{aligned} \tag{6}$$

Based on equations (5), (6) and definitions of χ^t_{K,Δ^I} and Δ^I,

$$\begin{aligned} & \Delta_b^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^I}) \\ = \ & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) \\ & + \frac{\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{k\in K} \Delta_k^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)} \big[\chi^t_{K,\Gamma^{\Omega,\Xi}}(\xi_K) - \sum_{k\in K} \Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m)\big] \\ = \ & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) + \frac{\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{k\in K} \Delta_k^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)} \big[\chi^t(\xi) \\ & \quad - \sum_{k\in\mathbb{P}\setminus K} \Delta_k^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) - \sum_{k\in K} \Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m)\big] \\ = \ & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) \\ & + \frac{\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{k\in K} \Delta_k^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)} \big[\sum_{k\in K} \Delta_k^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ & \quad - \sum_{k\in K} \Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m)\big] \\ & \textbf{(MOEIS of } \Delta^I\textbf{)} \\ = \ & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) \\ & + \frac{\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{k\in K} \Delta_k^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)} \Big[\frac{\sum_{k\in K} \Delta_k^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{p\in\mathbb{P}} \Omega(p)} \big[\chi^t(\xi) \\ & \quad - \sum_{p\in\mathbb{P}} \Delta_{\mathbb{P}}^t(\mathbb{P},\xi,\mathbb{X}^m)\big]\Big] \\ = \ & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) \\ & + \frac{\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)}{\sum_{p\in\mathbb{P}} \Omega(p)} \big[\chi^t(\xi) - \sum_{p\in\mathbb{P}} \Gamma_{\mathbb{P}}^t(\mathbb{P},\xi,\mathbb{X}^m)\big] \\ = \ & \Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) \end{aligned}$$

for every $b \in K$ and for every $t \in \mathbb{M}_m$. ∎

Theorem 2: On $\mathbb{MMC}^*$, the MMOIR is the only rule fitting MOIS and MORCON.

Proof of Theorem 2: By Lemma 4, the rule Δ^I fits MORCON. Clearly, the rule Δ^I fits MOIS.

To present the uniqueness, suppose that ρ fits MOIS and MORCON. By MOIS and MORCON of ρ, it is easy to clarify that ρ also fits MOIS, thus we omit it. Let $(\mathbb{P},\xi,\mathbb{X}^m) \in \mathbb{MMC}^*$, Ω be weight function for participating entities and Ξ be weight function for participation levels. By MOIS of ρ, $\rho(\mathbb{P},\xi,\mathbb{X}^m) = \Delta^{\Omega,\Xi}(\mathbb{P},\xi,\mathbb{X}^m)$ if $|\mathbb{P}| \leq 2$. The situation $|\mathbb{P}| > 2$: Let $b \in \mathbb{P}$, $t \in \mathbb{M}_m$ and $K = \{b,p\}$ with $p \in \mathbb{P} \setminus \{b\}$.

$$\begin{aligned} & \rho_b^t(\mathbb{P},\xi,\mathbb{X}^m) - \Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = \ & \rho_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho}) - \Delta_b^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^I}) \\ & \textbf{(MORCON of } \Delta^{I,t} \textbf{ and } \rho\textbf{)} \\ = \ & \Delta_b^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\rho}) - \Delta_b^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^I}). \\ & \textbf{(MOIS of } \rho\textbf{)} \end{aligned} \tag{7}$$

Similar to equation (2)

$$\Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho}) = \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) = \Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\Delta^I}). \tag{8}$$

By equations (7) and (8),

$$\begin{array}{rl} & \rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Delta_b^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\rho})-\Delta_b^{I,t}(K,\xi_K,\mathbb{X}^m_{K,\Delta^I,}) \\ = & \frac{\Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho})}{\Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho})+\Delta_p^t(K,\xi_K,\mathbb{X}^m_{K,\rho})}\left[\chi^t_{K,\rho}(\xi_K)-\chi^t_{K,\Delta^I}(\xi_K)\right] \\ = & \frac{\Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho})}{\Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho})+\Delta_p^t(K,\xi_K,\mathbb{X}^m_{K,\rho})}\big[\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)+\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m) \\ & \qquad -\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_p^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)\big]. \end{array}$$

Thus,

$$\begin{array}{rl} & \Delta_p^t(K,\xi_K,\mathbb{X}^m_{K,\rho})\cdot\left[\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = & \Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho})\cdot\left[\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_p^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)\right]. \end{array}$$

By MOEIS of Δ^I and ρ,

$$\begin{array}{rl} & \left[\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)\right]\cdot\sum\limits_{p\in\mathbb{P}}\Delta_p^t(K,\xi_K,\mathbb{X}^m_{K,\rho}) \\ = & \Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho})\cdot\sum\limits_{p\in\mathbb{P}}\left[\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_p^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = & \Delta_b^t(K,\xi_K,\mathbb{X}^m_{K,\rho})\cdot\left[\chi^t(\xi)-\chi^t(\xi)\right] \\ = & 0. \end{array}$$

Hence, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m)$ for every $b\in\mathbb{P}$ and for every $t\in\mathbb{M}_m$. ■

In the following some examples are exhibited to display that every of the properties applied in Theorem 2 is independent of the rest of properties.

Example 6: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}^*$, for every $t\in\mathbb{M}_m$ and for every participating entity $b\in\mathbb{P}$,

$$\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\begin{cases}\Delta_b^{I,t}(\mathbb{P},\xi,\mathbb{X}^m) & \text{if } |\mathbb{P}|\le 2,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, ρ fits MOIS, but it does not fit MORCON.

Example 7: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}^*$, for every $t\in\mathbb{M}_m$ and for every participating entity $b\in\mathbb{P}$, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=0$. Clearly, ρ fits MORCON, but it does not fit MOIS.

In the following, an instance is provide to present **(*)** how the new rules would distribute effects differently than the previous rules and **(**)** differently from each other.

Example 8: Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$ with $\mathbb{P}=\{a,b,c\}$, $m=2$, $\xi=(2,1,1)$, $\mathbb{L}_a=\{0,1_a,2_a\}$, $\mathbb{L}_b=\{0,1_b\}$, $\mathbb{L}_c=\{0,1_c\}$, $\Omega(a)=3$, $\Omega(b)=2$, $\Omega(c)=4$, $\Xi(1_a)=2$, $\Xi(2_a)=3$, $\Xi(1_b)=6$, $\Xi(1_c)=5$.

Further, let $\chi^1(2,1,1)=6$, $\chi^1(1,1,1)=8$, $\chi^1(2,1,0)=4$, $\chi^1(2,0,1)=3$, $\chi^1(2,0,0)=10$, $\chi^1(1,1,0)=4$, $\chi^1(1,0,1)=-5$, $\chi^1(0,1,1)=5$, $\chi^1(1,0,0)=-2$, $\chi^1(0,1,0)=3$, $\chi^1(0,0,1)=-4$, $\chi^2(2,1,1)=10$, $\chi^2(1,1,1)=4$, $\chi^2(2,1,0)=6$, $\chi^2(2,0,1)=7$, $\chi^2(2,0,0)=5$, $\chi^2(1,1,0)=-4$, $\chi^2(1,0,1)=5$, $\chi^2(0,1,1)=4$, $\chi^2(1,0,0)=8$, $\chi^2(0,1,0)=-3$, $\chi^2(0,0,1)=4$ and $\chi^1(0,0,0)=0=\chi^2(0,0,0)$. By Definitions 1, 2, 6,

$$\begin{array}{rclrcl}
\overline{\Delta_a^1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{5}{3}, & \overline{\Delta_b^1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{8}{3},\\
\overline{\Delta_c^1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{5}{3}, & \overline{\Delta_a^2}(\mathbb{P},\xi,\mathbb{X}^m) &=& 1,\\
\overline{\Delta_b^2}(\mathbb{P},\xi,\mathbb{X}^m) &=& 4, & \overline{\Delta_c^2}(\mathbb{P},\xi,\mathbb{X}^m) &=& 5,\\
\Delta_a^{\Omega,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{15}{9}, & \Delta_b^{\Omega,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{25}{9},\\
\Delta_c^{\Omega,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{14}{9}, & \Delta_a^{\Omega,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& 1,\\
\Delta_b^{\Omega,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{33}{9}, & \Delta_c^{\Omega,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{48}{9},\\
\Delta_a^{\Xi,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{-10}{3}, & \Delta_b^{\Xi,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{26}{3},\\
\Delta_c^{\Xi,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{2}{3}, & \Delta_a^{\Xi,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{-28}{3},\\
\Delta_b^{\Xi,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{26}{3}, & \Delta_c^{\Xi,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{32}{3},\\
\Delta_a^{\Omega,\Xi,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{-30}{9}, & \Delta_b^{\Omega,\Xi,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{106}{9},\\
\Delta_c^{\Omega,\Xi,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{-22}{9}, & \Delta_a^{\Omega,\Xi,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{-84}{9},\\
\Delta_b^{\Omega,\Xi,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{106}{9}, & \Delta_c^{\Omega,\Xi,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{68}{9},\\
\Delta_a^{I,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{12}{7}, & \Delta_b^{I,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{18}{7},\\
\Delta_c^{I,1}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{12}{7}, & \Delta_a^{I,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& 0,\\
\Delta_b^{I,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{30}{7}, & \Delta_c^{I,2}(\mathbb{P},\xi,\mathbb{X}^m) &=& \frac{40}{7}.
\end{array}$$

IV. BALANCE REQUIREMENT

In this section, related balance properties would be proposed to characterized the rules introduced in this study.

Definition 8:

- A rule ρ satisfies the **multi-objective balance (MOBAL)** requirement if

  $$\begin{array}{rl} & \rho_a^t(\mathbb{P},\xi,\mathbb{X}^m)-\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Delta_a^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m) \end{array}$$

  for every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, for every $t\in\mathbb{M}_m$, and for every $a,b\in\mathbb{P}$.
- The multi-objective balance requirement asserts that the disparity in remuneration allocated to any two participating entities should coincide with the difference between their respective minimal marginal level losses under any multi-objective multi-choice circumstance.

Lemma 5: The rule satisfies $\overline{\Delta}$ satisfies MOBAL on $\mathbb{MMC}$.

Proof: Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$. For every $a,b\in\mathbb{P}$,

$$\begin{array}{rl} & \overline{\Delta_a^t}(\mathbb{P},\xi,\mathbb{X}^m)-\overline{\Delta_b^t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Delta_a^t(\mathbb{P},\xi,\mathbb{X}^m)+\frac{1}{|\mathbb{P}|}\big[\chi^t(\xi)-\sum\limits_{k\in\mathbb{P}}\Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m)\big] \\ & -\Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m)+\frac{1}{|\mathbb{P}|}\big[\chi^t(\xi)-\sum\limits_{k\in\mathbb{P}}\Delta_k^t(\mathbb{P},\xi,\mathbb{X}^m)\big] \\ = & \Delta_a^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m). \end{array} \tag{9}$$

Hence, the rule satisfies $\overline{\Delta}$ satisfies MOBAL on $\mathbb{MMC}$. ■

Theorem 3: On $\mathbb{MMC}$, the MUEIL $\overline{\Delta}$ is the unique rule fitting MOEIS and MOBAL.

Proof of Theorem 3: By Lemmas 1 and 5, the rules $\overline{\Delta}$ fits MOEIS and MOBAL respectively.

To present the uniqueness, suppose that ρ fits MOEIS and MOBAL. Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, Ω be weight function for participating entities and Ξ be weight function for participation levels. By MOEIS of ρ, $\rho(\mathbb{P},\xi,\mathbb{X}^m)=\overline{\Delta}(\mathbb{P},\xi,\mathbb{X}^m)$ if $|\mathbb{P}|\le 1$. The situation $|\mathbb{P}|\ge 2$: Let $b\in\mathbb{P}$, $t\in\mathbb{M}_m$ and $K=\{b,p\}$ with $p\in\mathbb{P}\setminus\{b\}$. By applying MOBAL of ρ,

$$\begin{array}{rl} & \rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_p^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \overline{\Delta_b^t}(\mathbb{P},\xi,\mathbb{X}^m)-\overline{\Delta_p^t}(\mathbb{P},\xi,\mathbb{X}^m). \end{array} \tag{10}$$

By MOEIS of Δ^{Ξ} and ρ,

$$\begin{array}{ll} & |\mathbb{P}|\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\chi^t(\xi) \\ = & |\mathbb{P}|\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\sum_{p\in\mathbb{P}}\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \sum_{p\in\mathbb{P}}\left[\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = & \sum_{p\in\mathbb{P}}\left[\overline{\Delta_b^t}(\mathbb{P},\xi,\mathbb{X}^m)-\overline{\Delta_p^t}(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = & |\mathbb{P}|\overline{\Delta_b^t}(\mathbb{P},\xi,\mathbb{X}^m)-\sum_{p\in\mathbb{P}}\overline{\Delta_p^t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = & |\mathbb{P}|\overline{\Delta_b^t}(\mathbb{P},\xi,\mathbb{X}^m)-\chi^t(\xi). \end{array}$$

Hence, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\overline{\Delta_b^t}(\mathbb{P},\xi,\mathbb{X}^m)$ for every $b\in\mathbb{P}$ and for every $t\in\mathbb{M}_m$. ■

In the following some instances are exhibited to display that every of the requirements applied in Theorem 3 is independent of the rest of requirements.

Example 9: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t\in\mathbb{M}_m$ and for every participating entity $b\in\mathbb{P}$, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\frac{\chi^t(\xi)}{|\mathbb{P}|}$. Clearly, ρ fits MOEIS, but it does not fit MOBAL.

Example 10: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t\in\mathbb{M}_m$ and for every participating entity $b\in\mathbb{P}$, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\Delta_b^t(\mathbb{P},\xi,\mathbb{X}^m)$. Clearly, ρ fits MOBAL, but it does not fit MOEIS.

In the context of multi-objective considerations, the same participation level adopted by a single participating entity may yield varying effects relative to different objectives. Therefore, the subsequent requirement is to incorporate weighted considerations into the previously proposed multi-objective balance requirement. Specifically, the disparity in remuneration allocated to any two participating entities should coincide with the difference between their respective weighted minimal marginal level losses.

Definition 9:

- A rule ρ satisfies the **weighted multi-objective balance (WMOBAL)** requirement if

$$\begin{array}{ll} & \rho_a^t(\mathbb{P},\xi,\mathbb{X}^m)-\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Gamma_a^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m) \end{array}$$

  for every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, for every $t\in\mathbb{M}_m$, and for every $a,b\in\mathbb{P}$.
- The multi-objective balance requirement asserts that the disparity in remuneration allocated to any two participating entities should coincide with the difference between their respective weighted minimal marginal level losses.

Lemma 6: The rule satisfies Δ^{Ξ} satisfies WMOBAL on $\mathbb{MMC}$.

Proof: Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$. For every $a,b\in\mathbb{P}$,

$$\begin{array}{ll} & \Delta_a^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Gamma_a^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)+\frac{1}{|\mathbb{P}|}\left[\chi^t(\xi)-\sum_{k\in\mathbb{P}}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ & -\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)+\frac{1}{|\mathbb{P}|}\left[\chi^t(\xi)-\sum_{k\in\mathbb{P}}\Gamma_k^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = & \Gamma_a^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m). \end{array} \tag{11}$$

Hence, the rule satisfies Δ^{Ξ} satisfies WMOBAL on $\mathbb{MMC}$. ■

Theorem 4: On $\mathbb{MMC}$, the MUEIL Δ^{Ξ} is the unique rule fitting MOEIS and WMOBAL.

Proof of Theorem 3: By Lemmas 1 and 5, the rules Δ^{Ξ} fits MOEIS and WMOBAL respectively.

To present the uniqueness, suppose that ρ fits MOEIS and WMOBAL. Let $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, Ω be weight function for participating entities and Ξ be weight function for participation levels. By MOEIS of ρ, $\rho(\mathbb{P},\xi,\mathbb{X}^m)=\Delta^{\Xi}(\mathbb{P},\xi,\mathbb{X}^m)$ if $|\mathbb{P}|\leq 1$. The situation $|\mathbb{P}|\geq 2$: Let $b\in\mathbb{P}$, $t\in\mathbb{M}_m$ and $K=\{b,p\}$ with $p\in\mathbb{P}\setminus\{b\}$. By applying WMOBAL of ρ,

$$\begin{array}{ll} & \rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\Gamma_p^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \Delta_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_p^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m). \end{array} \tag{12}$$

By MOEIS of Δ^{Ξ} and ρ,

$$\begin{array}{ll} & |\mathbb{P}|\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\chi^t(\xi) \\ = & |\mathbb{P}|\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\sum_{p\in\mathbb{P}}\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m) \\ = & \sum_{p\in\mathbb{P}}\left[\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)-\rho_p^t(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = & \sum_{p\in\mathbb{P}}\left[\Delta_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\Delta_p^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)\right] \\ = & |\mathbb{P}|\Delta_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\sum_{p\in\mathbb{P}}\Delta_p^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m) \\ = & |\mathbb{P}|\Delta_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)-\chi^t(\xi). \end{array}$$

Hence, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\Delta_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)$ for every $b\in\mathbb{P}$ and for every $t\in\mathbb{M}_m$. ■

In the following some instances are exhibited to display that every of the requirements applied in Theorem 3 is independent of the rest of requirements.

Example 11: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t\in\mathbb{M}_m$ and for every participating entity $b\in\mathbb{P}$, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\frac{\chi^t(\xi)}{|\mathbb{P}|}$. Clearly, ρ fits MOEIS, but it does not fit WMOBAL.

Example 12: We focus on the rule ρ as follows. For every $(\mathbb{P},\xi,\mathbb{X}^m)\in\mathbb{MMC}$, for every weight function for participating entities Ω, for every weight function for participation levels Ξ, for every $t\in\mathbb{M}_m$ and for every participating entity $b\in\mathbb{P}$, $\rho_b^t(\mathbb{P},\xi,\mathbb{X}^m)=\Gamma_b^{\Xi,t}(\mathbb{P},\xi,\mathbb{X}^m)$. Clearly, ρ fits WMOBAL, but it does not fit MOEIS.

Remark 1: Based on Definitions 2, 6, 8, 9, it is easy to check that the 1-MWEIL, the BWEIL and the MMOIR violates the multi-objective balance requirement and the weighted multi-objective balance requirement.

V. Conclusions

In contrast to existing studies, this research defines different weighted functions for participating entities and their respective participation levels simultaneously in the context of multi-objective multi-choice scenarios, thereby proposing loss evaluating rules such as the MUEIL, the 1-MWEIL, the 2-MWEIL, and the BWEIL, along with its related axiomatic processes. Unlike subjective weighted functions, this study naturally utilizes minimal marginal level-losses instead of traditional weights, introducing the MMOIR and its related axiomatic processes within the framework of multi-objective multi-choice settings. It is necessary to compare and analyze the findings of existing literature with the results of this study.

1) Traditional evaluating rules mostly focus on participating patterns of either participation or non-participation for participating entities.

2) The MUEIL, the 1-MWEIL, the 2-MWEIL, the BWEIL, and the MMOIR and related axiomatic processes proposed in this study have not been appeared in previous research literature.
   - Under the evaluating notions of the MUEIL and the 2-MWEIL, after evaluating the different minimal marginal level-losses of all participating entities, the remaining loss effects are shared among all participating entities.
   - Under the evaluating notions of the 1-MWEIL and the BWEIL, after evaluating the different minimal marginal level-losses of all participating entities, the remaining loss benefits are allocated based on the relative weighted proportions of each participating entity.
   - The evaluating notions of the 2-MWEIL and the BWEIL consider weights for participation levels in evaluating the different minimal marginal level-losses for all participating entities, which is not considered in the MUEIL and the 1-MWEIL.
   - Participating entities and their participation levels are crucial in multi-objective multi-choice circumstances. Therefore, weights should be applied to both participating entities and their participation levels. Under the evaluating notion of the BWEIL, the minimal weighted marginal level-losses of participating entities are first evaluated, and then the remaining losses are allocated based on the relative weighted proportions of participation factors.
   - However, using weighted functions for evaluating may lack naturalness. Therefore, in the evaluating notion of the MMOIR, the minimal marginal level-losses of all participating entities are first evaluated, and then the remaining losses are borne based on the corresponding proportions due to the minimal marginal level-losses.

The multi-objective balance requirement and the weighted multi-objective balance requirement, as proposed in Definitions 8 and 9, can be regarded as two distinct types of symmetry requirements. The multi-objective balance requirement implies that the disparity in remuneration allocated to any two participating entities should be symmetric with the difference between their respective minimal marginal level losses. Conversely, the weighted multi-objective balance requirement suggests that the disparity in remuneration between any two participating entities should be symmetric with the difference between their respective weighted minimal marginal level losses.

The loss evaluating rules proposed in this study have several advantages.

- Traditional evaluating criteria under traditional circumstances often consider the non-participation or universal participation of all participating entities. However, this study takes into account that all participating entities can operate at different participation levels.
- In some studies on evaluating rules under multi-choice circumstances, although the participation levels of participating entities are considered, the evaluation is based on related evaluating results of specific participating entities at specific participation levels. In this study, however, the overall impact generated by all participation levels of each participating entity is observed.
- To reflect real-world scenarios, the BWEIL conducts loss evaluation based on the weighting of participating entities and their participation levels simultaneously. Furthermore, considering potential issues regarding the fairness or legitimacy of weighted functions, the MMOIR utilizes relative minimal marginal level-losses instead of weighting.

Nevertheless, the rules proposed in this study have some limitations. As emphasized in the advantages, each participating entity has different participation levels. Although it is possible to consider the overall effect of all participation levels of each participating entity, it is not possible to evaluate the effect of specific participating entities at specific participation levels. Future research should explore alternative rules that simultaneously consider both overall effect and relative effect of a specific participation level.

The findings of this study also motivate further research.

- In addition to the EANSC, alternative rules for evaluating losses under multi-objective multi-choice scenarios derived from existing evaluating rules can be considered.

Readers are encouraged to conduct further research based on the aforementioned motivations.

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