

Solving System of ODEs and Analyzing Chemical Reactions

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Abstract—The Aboodh transform is a mathematical technique used in the fields of mathematical analysis and signal processing. The definition of Aboodh transform (AT), fundamental properties related to the convolution, linearity, AT for derivatives, dualities to other transforms, and others are discussed in this study. This article focuses on various features and the applications of the AT in solving specific classes of ordinary differential equation (ODE) systems. In the application section, the concentration of chemical reactants is computed in a series of reactions to chemicals using a physical mathematical problem. To address this problem, ODEs are used to create a mathematical model, and the AT is applied to obtain the solution and analyze the results. The findings of this study are presented through tables and graphs, demonstrating the accuracy and efficiency of solving ODEs using the AT.

Index Terms—Aboodh transform, Laplace transform, differential equation, system of differential equation.

I. INTRODUCTION

IN the subject of mathematics, and in its many applications, differential equations play a crucial role. These equations are essential for modeling and comprehending dynamic processes in physics, engineering, biology, economics, and other fields because they serve as a fundamental language for expressing how quantities change in relation to one another [1], [2], [3]. Differential equations offer a potent tool for revealing the underlying principles regulating complicated events, whether it be through modelling the behavior of financial markets, forecasting the trajectory of a rocket, or studying population increase [4], [5], [6]. They help scientists and engineers make educated judgments, optimize designs, and resolve challenging real-world challenges by bridging the gap between theory and practice. Differential equations are a cornerstone in the quest for knowledge and innovation since, in essence, they are studied and used to drive advances in science and technology.

Diverse mathematical strategies that are used to solve equations involving derivatives are included in methods for solving differential equations. Analytical and numerical approaches can be divided into two basic categories. By modifying the equation algebraically, analytical techniques like variable separation [7], [8], substitution, and integrating factors [9], [10], [11] seek to arrive at precise answers. They are especially beneficial for straightforward, well-organized

problems. By dividing the issue into discrete steps and employing iterative algorithms, numerical methods, including Euler's method [12], [13], [14], finite difference methods [15], [16], and Runge-Kutta methods [17], [18], [19] approximate answers. When working with complicated, non-linear, or time-dependent systems, these numerical techniques are essential.

Additionally, integral transformations powerful mathematical tools that are essential in many fields of science and engineering are used to solve differential equations. These changes entail transforming a given function into an alternative representation, which frequently simplifies complex issues and offers new perspectives on several fields. The Fourier transform [20], [21], [22], one of the most well-known integral transformations, decomposes a function into a sum of sinusoidal components, making it useful in signal processing and spectrum analysis. Another crucial transformation for the analysis of linear time-invariant systems and the solution of differential equations is the Laplace transform [23], [24]. The Sumudu-transform [25], [26], [27] is frequently used in control theory and discrete-time signal processing. There are many other transforms that have been introduced in literature with various applications in applied mathematics, such as Elzaki transform [28], [29], [30], ARA transform [31], [32], Formable transform [33] and Mohand transform [34], [35]. Moreover, double integral transforms have been shown, such as double Laplace [36], [37], double Formable [38], [39], double ARA and others [40].

The Aboodh integral transform [41], [42] is another mathematical method used to the study of mathematics and signal processing. This transform, which bears Professor Aboodh Alkaabi's name, is vital in moving functions from their original domain into a new domain where they might be simpler to comprehend or solve. When analyzing complex integrals and solving differential equations, it is especially helpful. A significant tool in many scientific and engineering applications, the AT can make it easier to handle functions with complex mathematical features. In conclusion, these essential changes, among others, are crucial for modeling and problem-solving across a wide range of fields, enabling deeper comprehension and creative responses to challenging problems.

The main goal of this study is to present the definition of AT, its main properties, and its values for its basic functions. Moreover, we discuss the solutions to various types of ODEs and the systems, using the proposed transform with an applicable approach. The outcomes gained are utilized to study the concentration of reactants in the chemicals using a physical chemistry problem. A model is discussed and solved by AT. We use Python software to generate the numerical results and sketch the figures.

This article is organized as follows: In Section 2, we

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introduce the main definitions and properties of AT. The application of AT to solve systems of ODEs is discussed in Section 3, we move on physical chemical application in Section 4, and we show the results in Section 5 as a conclusion.

II. PRELIMINARIES

In this section, some fundamental definitions and properties of AT are presented.

Definition 2.1: [41] The Aboodh transform (AT) is defined by the integral equation

$$A[\psi(\eta)] = \frac{1}{\delta} \int_0^{\infty} e^{-\delta\eta} \psi(\eta) d\eta = \Psi(\delta), \quad \eta > 0. \quad (1)$$

Definition 2.2: [41] If $\Psi(\delta) = A[\psi(\eta)]$ then $\psi(\eta)$ is the inverse of AT of $\Psi(\delta)$, can be expressed as

$$\psi(\eta) = A^{-1}[\Psi(\delta)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \delta e^{-\eta\delta} \Psi(\delta) d\delta, \quad c \in \mathbb{R}, \quad (2)$$

where A^{-1} is the operator of inverse AT.

A. Some Properties of AT

[41] Herein we present the fundamental properties of AT.

i. Linearity property of AT.

Let $A[\psi_1(\eta)] = \Psi_1(\delta)$ and $A[\psi_2(\eta)] = \Psi_2(\delta)$ then

$$A[b\psi_1(\eta) + c\psi_2(\eta)] = b\Psi_1(\delta) + c\Psi_2(\delta),$$

where b and c are arbitrary constants.

Proof: From the definition of AT, we have

$$\begin{aligned} A[b\psi_1(\eta) + c\psi_2(\eta)] &= \frac{1}{\delta} \int_0^{\infty} (b\psi_1(\eta) + c\psi_2(\eta)) e^{-\delta\eta} d\eta \\ &= b \left(\frac{1}{\delta} \int_0^{\infty} \psi_1(\eta) e^{-\delta\eta} d\eta \right) \\ &\quad + c \left(\frac{1}{\delta} \int_0^{\infty} \psi_2(\eta) e^{-\delta\eta} d\eta \right) \\ &= b A[\psi_1(\eta)] + c A[\psi_2(\eta)] \\ &= b\Psi_1(\delta) + c\Psi_2(\delta). \end{aligned} \quad (3)$$

Moreover, the inverse AT is linear. If

$$A^{-1}[\Psi_1(\delta)] = \psi_1(\eta) \quad (4)$$

and

$$A^{-1}[\Psi_2(\delta)] = \psi_2(\eta).$$

Then,

$$\begin{aligned} A^{-1}[b\Psi_1(\delta) + c\Psi_2(\delta)] &= bA^{-1}[\Psi_1(\delta)] + cA^{-1}[\Psi_2(\delta)] \\ &= b\psi_1(\eta) + c\psi_2(\eta). \end{aligned} \quad (5)$$

ii. Scaling property of AT

If AT of a function $\psi(\eta)$ is $\Psi(\delta)$, then AT of $\psi(k\eta)$ is defined by $\frac{1}{k^2} \left(\Psi\left(\frac{\delta}{k}\right) \right)$, where k is any real number.

Proof: The definition of AT, implies

$$A[\psi(k\eta)] = \frac{1}{\delta} \int_0^{\infty} \psi(k\eta) e^{-\delta\eta} d\eta. \quad (6)$$

Putting $k\eta = p$, then $k d\eta = dp$ in the equation (6), we have

$$\begin{aligned} A[\psi(k\eta)] &= \frac{1}{\delta} \left(\int_0^{\infty} \psi(p) e^{-\frac{\delta}{k}p} \frac{dp}{k} \right) \\ &= \frac{1}{k^2} \left(\frac{k}{\delta} \int_0^{\infty} \psi(p) e^{-\frac{\delta}{k}p} dp \right) \\ &= \frac{1}{k^2} \left(\Psi\left(\frac{\delta}{k}\right) \right). \end{aligned} \quad (7)$$

iii. Shifting property of AT

If AT of a function $\psi(\eta)$ is $\Psi(\delta)$, then AT of the function $e^{k\eta}\psi(\eta)$ is given by

$$\frac{(\delta - K)}{\delta} \Psi(\delta - k),$$

where $k \in \mathbb{R}$.

Proof: By the definition of AT, we get

$$\begin{aligned} A[e^{k\eta}\psi(\eta)] &= A[\psi(\eta)] = \frac{1}{\delta} \int_0^{\infty} e^{-\delta\eta} \psi(\eta) d\eta \\ &= \frac{1}{\delta} \int_0^{\infty} e^{-(\delta-k)\eta} \psi(\eta) d\eta \\ &= \frac{(\delta - K)}{\delta} \frac{1}{(\delta - k)} \int_0^{\infty} e^{-(\delta-k)\eta} \psi(\eta) d\eta \\ &= \frac{(\delta - K)}{\delta} \Psi(\delta - k). \end{aligned} \quad (8)$$

The AT of some functions are presented in Table I, below.

TABLE I
AT OF SOME ELEMENTARY FUNCTIONS

Function $\psi(\eta)$	$A[\psi(\eta)] = \Psi(\delta)$
1	$\frac{1}{\delta^2}$
η	$\frac{1}{\delta^3}$
η^2	$\frac{2}{\delta^4}$
$\eta^\alpha, \alpha > 0$	$\frac{\Gamma(\alpha+1)}{\delta^{\alpha+2}}$
$e^{a\eta}$	$\frac{1}{\delta^2 - a\delta}$
$\sin(\alpha\eta)$	$\frac{1}{\delta(\delta^2 + \alpha^2)}$
$\cos(\alpha\eta)$	$\frac{1}{\delta^2 + \alpha^2}$
$\sinh(\alpha\eta)$	$\frac{1}{\delta(\delta^2 - \alpha^2)}$
$\cosh(\alpha\eta)$	$\frac{1}{\delta^2 - \alpha^2}$

1) *Relation Between AT and Some Transforms:* This section presents the relation between AT and other popular transforms.

• Laplace transform

Assume that $L[\psi(\eta)] = \int_0^{\infty} e^{-\delta\eta} \psi(\eta) d\eta$ is the Laplace transform of $\psi(\eta)$, then $A[\psi(\eta)] = \frac{1}{\delta} L[\psi(\eta)]$.

Proof:

$$\begin{aligned} A[\psi(\eta)] &= \frac{1}{\delta} \left(\int_0^{\infty} e^{-\delta\eta} \psi(\eta) d\eta \right) \\ &= \frac{1}{\delta} L[\psi(\eta)] \\ &= \frac{1}{\delta} \Psi(\delta). \end{aligned} \quad (9)$$

• Sumudu transform

If $S[\psi(\eta)] = \frac{1}{\delta} \int_0^\infty e^{-\frac{\eta}{\delta}} \psi(\eta) d\eta$ is the Sumudu transform of $\psi(\eta)$, then $A[\psi(\eta)] = \delta^2 S[\psi(\eta)]$.

Proof:

$$A[\psi(\eta)] = \frac{1}{\delta} \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta = \Psi(\delta). \quad (10)$$

Moreover

$$\begin{aligned} \Psi\left(\frac{1}{\delta}\right) &= \delta \int_0^\infty e^{-\frac{\eta}{\delta}} \psi\left(\frac{1}{\delta}\right) d\eta \\ &= \delta^2 \frac{1}{\delta} \int_0^\infty e^{-\frac{\eta}{\delta}} \psi\left(\frac{1}{\delta}\right) d\eta \\ &= \delta^2 \Psi\left(\frac{1}{\delta}\right). \end{aligned}$$

• Elzaki transform

If $E[\psi(\eta)] = \delta \int_0^\infty e^{-\frac{\eta}{\delta}} \psi(\eta) d\eta$ is the Elzaki transform of $\psi(\eta)$, then $A[\psi(\frac{1}{\delta})] = E[\psi(\eta)]$.

Proof:

$$A[\psi(\eta)] = \frac{1}{\delta} \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta = \Psi(\delta). \quad (11)$$

Moreover

$$\Psi\left(\frac{1}{\delta}\right) = \delta \int_0^\infty e^{-\frac{\eta}{\delta}} \psi\left(\frac{1}{\delta}\right) d\eta = E[\psi(\eta)].$$

• Mohand transform

If $M[\psi(\eta)] = \delta^2 \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta$ is the Mohand transform of $\psi(\eta)$, then: $A[\psi(\eta)] = \frac{1}{\delta^3} M[\psi(\eta)]$.

Proof:

$$A[\psi(\eta)] = \frac{1}{\delta} \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta = \Psi(\delta). \quad (12)$$

Moreover,

$$A[\psi(\eta)] = \frac{1}{\delta^3} \delta^2 \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta = \frac{1}{\delta^3} M[\psi(\eta)].$$

• Formable transform

If $B(\delta, u) = \delta \int_0^\infty e^{-\delta\eta} \psi(u\eta) d\eta$ is the formable transform of $\psi(u\eta)$, then $A[\psi(\eta)] = \frac{1}{\delta^2} B(\delta, 1)$.

Proof:

$$\begin{aligned} A[\psi(\eta)] &= \frac{1}{\delta} \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta \\ &= \frac{1}{\delta^2} \left(\delta \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta \right) \\ &= \frac{1}{\delta^2} B(\delta, 1). \end{aligned} \quad (13)$$

B. SAT for Derivatives

If $\Psi(\delta) = A[\psi(\eta)]$, then

i.

$$A[\psi'(\eta)] = \delta\Psi(\delta) - \frac{\psi'(0)}{\delta}. \quad (14)$$

ii.

$$A[\psi''(\eta)] = \delta^2\Psi(\delta) - \psi(0) - \frac{\psi'(0)}{\delta}. \quad (15)$$

iii.

$$A[\psi^{(n)}(\eta)] = \delta^{(n)}\Psi(\delta) \sum_{k=0}^{n-1} \frac{\psi^{(k)}(0)}{\delta^{2-n+k}}. \quad (16)$$

The proof of (i), can be obtained as:

$$A[\psi'(\eta)] = \frac{1}{\delta} \int_0^\infty e^{-\delta\eta} \psi'(\eta) d\eta. \quad (17)$$

Using integration by parts, we have:

$u = e^{-\delta\eta}$ and $dv = \psi'(\eta)$ then $du = -\delta e^{-\delta\eta}$, $v = \psi(\eta)$.

Then, the equation 17 becomes

$$\begin{aligned} A[\psi'(\eta)] &= \frac{1}{\delta} e^{-\delta\eta} \psi(\eta) \Big|_0^\infty + \delta \int_0^\infty e^{-\delta\eta} \psi(\eta) d\eta \\ &= \frac{1}{\delta} (0 - \psi(0)) + \delta A[\psi(\eta)] \\ &= A[\psi(\eta)] - \frac{1}{\delta} \psi(0). \end{aligned} \quad (18)$$

The proof of (ii): we have $A[\psi''(\eta)] = A[(\psi'(\eta))']$ using part (i), we get

$$\begin{aligned} A[(\psi'(\eta))'] &= \delta A[\psi'(\eta)] - \frac{1}{\delta} \psi'(0) \\ &= \delta \left[\delta A[\psi(\eta)] - \frac{1}{\delta} \psi(0) \right] - \frac{1}{\delta} \psi'(0) \\ &= \delta^2 A[\psi(\eta)] - \frac{1}{\delta} \psi'(0) - \psi(0). \end{aligned} \quad (19)$$

The proof of (iii), can be obtained by mathematical induction, as $n = 1$, we get part (i).

Now suppose its true for $n = k$, then

$$A[\psi^{(k)}(\eta)] = \delta^k \Psi(\delta) - \frac{\psi(0)}{\delta^{2-k}} - \frac{\psi'(0)}{\delta^{3-k}} - \dots - \frac{\psi^{(k-1)}(0)}{\delta}. \quad (20)$$

To show its true for $n = k + 1$, use the formula in (i). $A[\psi^{(k+1)}(\eta)] = A[\psi^{(k)}(\eta)]'$

$$\begin{aligned} A[\psi^{(k+1)}] &= A\left[\left(\psi^{(k)}(\eta)\right)'\right] \\ &= \delta^{k+1} \Psi(\delta) - \frac{1}{\delta^{1-k}} \psi(0) - \frac{\psi'(0)}{\delta^{2-k}} \\ &\quad - \frac{\psi''(0)}{\delta^{3-k}} - \dots - \frac{\psi^{(k)}(0)}{\delta} \\ &= \delta^{(n)} \Psi(\delta) - \sum_{k=0}^{n-1} \frac{\psi^{(k)}(0)}{\delta^{2-n+k}}. \end{aligned} \quad (21)$$

III. AT FOR SOLVING SYSTEM OF ODES

Now we present AT for solving systems of ODEs. To do so, let's take the system of ODEs, below:

$$\begin{aligned} \frac{d\psi_1}{d\eta} &= b_{11}\psi_1(\eta) + b_{12}\psi_2(\eta) + b_{13}\psi_3(\eta) + \dots \\ &\quad + b_{1n}\psi_n(\eta) + \phi_1(\eta), \\ \frac{d\psi_2}{d\eta} &= b_{21}\psi_1(\eta) + b_{22}\psi_2(\eta) + b_{23}\psi_3(\eta) + \dots \\ &\quad + b_{2n}\psi_n(\eta) + \phi_2(\eta), \\ &\quad \vdots \\ \frac{d\psi_n}{d\eta} &= b_{n1}\psi_1(\eta) + b_{n2}\psi_2(\eta) + b_{n3}\psi_3(\eta) + \dots \\ &\quad + b_{nn}\psi_n(\eta) + \phi_n(\eta). \end{aligned} \quad (22)$$

with the initial conditions:

$$\psi_1(0) = r_1, \psi_2(0) = r_2, \dots, \psi_n(0) = r_n, \quad (23)$$

where $b_{11}, b_{12}, b_{13}, \dots, b_{nn}$ are constants, and $\psi_1(\eta), \psi_2(\eta), \dots, \psi_n(\eta)$ are unknown continuous functions, and $\phi_1(\eta), \phi_2(\eta), \dots, \phi_n(\eta)$ are continuous given functions.

The system (22) can be expressed in matrix form with (23) as

$$\frac{d\psi}{d\eta} = B\psi(\eta) + \phi(\eta), \text{ with } \psi(0) = R, \quad (24)$$

where

$$\frac{d\psi}{d\eta} = \begin{bmatrix} \frac{d\psi_1}{d\eta} \\ \frac{d\psi_2}{d\eta} \\ \vdots \\ \frac{d\psi_n}{d\eta} \end{bmatrix}, B = \begin{bmatrix} b_{11}, b_{12}, \dots, b_{1n} \\ b_{21}, b_{22}, \dots, b_{2n} \\ \vdots \\ b_{n1}, b_{n2}, \dots, b_{nn} \end{bmatrix},$$

$$\psi(\eta) = \begin{bmatrix} \psi_1(\eta) \\ \psi_2(\eta) \\ \vdots \\ \psi_n(\eta) \end{bmatrix}, \phi(\eta) = \begin{bmatrix} \phi_1(\eta) \\ \phi_2(\eta) \\ \vdots \\ \phi_n(\eta) \end{bmatrix}, \psi(0) = \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \\ \vdots \\ \psi_n(0) \end{bmatrix},$$

and

$$R = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}.$$

By applying AT to system (22), we have

$$\left. \begin{aligned} A\{\psi_1(\eta)'\} &= b_{11}A\{\psi_1(\eta)\} + b_{12}A\{\psi_2(\eta)\} + \dots \\ &\quad + b_{1n}A\{\psi_n(\eta)\} + A\{\phi_1(\eta)\}, \\ A\{\psi_2(\eta)'\} &= b_{21}A\{\psi_1(\eta)\} + b_{22}A\{\psi_2(\eta)\} \\ &\quad + \dots + b_{2n}A\{\psi_n(\eta)\} + A\{\phi_2(\eta)\}, \\ &\quad \vdots \\ A\{\psi_n(\eta)'\} &= b_{n1}A\{\psi_1(\eta)\} + b_{n2}A\{\psi_2(\eta)\} + \dots \\ &\quad + b_{nn}A\{\psi_n(\eta)\} + A\{\phi_n(\eta)\}. \end{aligned} \right\} (25)$$

Then, we get

$$\left. \begin{aligned} \delta A\{\psi_1(\eta)\} - \frac{1}{\delta}\psi_1(0) &= b_{11}A\{\psi_1(\eta)\} + b_{12}A\{\psi_2(\eta)\} \\ &\quad + \dots + b_{1n}A\{\psi_n(\eta)\} + A\{\phi_1(\eta)\}, \\ \delta A\{\psi_2(\eta)\} - \frac{1}{\delta}\psi_2(0) &= b_{21}A\{\psi_1(\eta)\} + b_{22}A\{\psi_2(\eta)\} \\ &\quad + \dots + b_{2n}A\{\psi_n(\eta)\} + A\{\phi_2(\eta)\}, \\ &\quad \vdots \\ \delta A\{\psi_n(\eta)\} - \frac{1}{\delta}\psi_n(0) &= b_{n1}A\{\psi_1(\eta)\} + b_{n2}A\{\psi_2(\eta)\} \\ &\quad + \dots + b_{nn}A\{\psi_n(\eta)\} + A\{\phi_n(\eta)\}. \end{aligned} \right\} (26)$$

Now, using the ICs (23), the system (26) becomes

$$\left. \begin{aligned} (\delta - b_{11})A\{\Psi_1(\eta)\} - b_{12}A\{\Psi_2(\eta)\} - \dots - b_{1n}A\{\Psi_n(\eta)\} \\ = A\{\psi_1(\eta)\} + \frac{r_1}{\delta}, \\ -b_{n1}A\{\Psi_1(\eta)\} + (\delta - b_{22})A\{\Psi_2(\eta)\} - \dots \\ - b_{2n}A\{\Psi_n(\eta)\} = A\{\psi_2(\eta)\} + \frac{r_2}{\delta}, \\ \vdots \\ -b_{n1}A\{\psi_1(\eta)\} - b_{n2}A\{\psi_2(\eta)\} - \dots \\ + (\delta - b_{nn})A\{\psi_n(\eta)\} = A\{\psi_n(\eta)\} \\ + \frac{r_n}{\delta}. \end{aligned} \right\} (27)$$

Cramer's rules are used now to solve system (25) as:

$$\begin{aligned} [\psi_1(\eta)] &= \frac{\begin{vmatrix} A\{\psi_1(\eta)\} + \frac{r_1}{\delta} & -b_{12} & \dots & -b_{1n} \\ A\{\psi_2(\eta)\} + \frac{r_2}{\delta} & (\delta - b_{22}) & \dots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A\{\psi_n(\eta)\} + \frac{r_n}{\delta} & -b_{n2} & \dots & (\delta - b_{nn}) \end{vmatrix}}{\begin{vmatrix} (\delta - b_{11}) & -b_{12} & \dots & -b_{1n} \\ -b_{21} & (\delta - b_{22}) & \dots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \dots & (\delta - b_{nn}) \end{vmatrix}}, \\ A[\psi_2(\eta)] &= \frac{\begin{vmatrix} (\delta - b_{11}) & A\{\psi_1(\eta)\} + \frac{r_1}{\delta} & \dots & -b_{1n} \\ -b_{21} & A\{\psi_2(\eta)\} + \frac{r_2}{\delta} & \dots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & A\{\psi_n(\eta)\} + \frac{r_n}{\delta} & \dots & (\delta - b_{nn}) \end{vmatrix}}{\begin{vmatrix} (\delta - b_{11}) & -b_{12} & \dots & -b_{1n} \\ -b_{21} & (\delta - b_{22}) & \dots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \dots & (\delta - b_{nn}) \end{vmatrix}}, \\ A[\psi_n(\eta)] &= \frac{\begin{vmatrix} (\delta - b_{11}) & -b_{1n} & \dots & A\{\psi_1(\eta)\} + \frac{r_1}{\delta} \\ -b_{21} & -b_{2n} & \dots & A\{\psi_2(\eta)\} + \frac{r_2}{\delta} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & (\delta - b_{nn}) & \dots & A\{\psi_n(\eta)\} + \frac{r_n}{\delta} \end{vmatrix}}{\begin{vmatrix} (\delta - b_{11}) & -b_{12} & \dots & -b_{1n} \\ -b_{21} & (\delta - b_{22}) & \dots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \dots & (\delta - b_{nn}) \end{vmatrix}}. \end{aligned}$$

Now applying the inverse AT of $A[\psi_1(\eta)], A[\psi_2(\eta)], \dots, A[\psi_n(\eta)]$, then we get the value of $\psi_1(\eta), \psi_2(\eta), \dots, \psi_n(\eta)$.

Example 3.1. Let us system of ODEs:

$$\left. \begin{aligned} \frac{d\psi_1}{d\eta} &= \psi_3(\eta), \\ \frac{d\psi_2}{d\eta} &= -\psi_3(\eta), \\ \frac{d\psi_3}{d\eta} &= -\psi_1(\eta) - \psi_2(\eta). \end{aligned} \right\} (28)$$

with ICs

$$\psi_1(0) = 0, \psi_2(0) = 1 \text{ and } \psi_3(0) = 0. \quad (29)$$

Sol. The system (28) with the ICs (29) can be written:

$$\frac{d\psi}{d\eta} = B\psi(\eta) + \phi(\eta), \text{ with } \psi(0) = R, \quad (30)$$

where

$$\frac{d\psi}{d\eta} = \begin{bmatrix} \frac{d\psi_1}{d\eta} \\ \frac{d\psi_2}{d\eta} \\ \frac{d\psi_3}{d\eta} \end{bmatrix}, H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}, \psi(\eta) = \begin{bmatrix} \psi_1(\eta) \\ \psi_2(\eta) \\ \psi_3(\eta) \end{bmatrix},$$

$$\phi(\eta) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \psi(0) = \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \\ \psi_3(0) \end{bmatrix}, \text{ and, } R = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

By applying AT to the system (28), we get:

$$\left. \begin{aligned} A[\psi'_1(t)] - A[\psi_3(t)] &= 0, \\ A[\psi'_2(t)] + A[\psi_3(t)] &= 0, \\ A[\psi_1(t)] + A[\psi_2(t)] + A[\psi'_3(t)] &= 0. \end{aligned} \right\} \quad (31)$$

Running AT to the system (31) and using the ICs (29)

$$\left. \begin{aligned} \delta A[\psi_1(\eta)] - \frac{1}{\delta}\psi_1(0) - A[\psi_3(\eta)] &= 0, \\ \delta A[\psi_2(\eta)] - \frac{1}{\delta}\psi_2(0) - A[\psi_3(\eta)] &= 0, \\ A[\psi_1(\eta)] + A[\psi_2(\eta)] + \delta A[\psi'_3(\eta)] - \frac{1}{\delta}\psi_3(0) &= 0. \end{aligned} \right\} \quad (32)$$

Simplifying the system (32), we obtain

$$\left. \begin{aligned} \delta A[\psi_1(\eta)] - A[\psi_3(\eta)] &= 0, \\ \delta A[\psi_2(\eta)] + A[\psi_3(\eta)] &= \frac{1}{\delta}, \\ A[\psi_1(\eta)] + A[\psi_2(\eta)] + \delta A[\psi_3(\eta)] &= 0. \end{aligned} \right\} \quad (33)$$

Using Cramer's rule to solve $A[\psi_1(\eta)]$, $A[\psi_2(\eta)]$ and $A[\psi_3(\eta)]$ on the system (33), we get

$$A[\psi_1(\eta)] = \frac{\begin{vmatrix} 0 & 0 & -1 \\ \frac{1}{\delta} & \delta & 1 \\ 0 & 1 & \delta \end{vmatrix}}{\begin{vmatrix} \delta & 0 & -1 \\ 0 & \delta & 1 \\ 1 & 1 & \delta \end{vmatrix}} = \frac{-1}{\delta^4}, \quad (34)$$

$$A[\psi_2(\eta)] = \frac{\begin{vmatrix} \delta & 0 & -1 \\ 0 & \frac{1}{\delta} & 1 \\ 1 & 0 & \delta \end{vmatrix}}{\begin{vmatrix} \delta & 0 & -1 \\ 0 & \delta & 1 \\ 1 & 1 & \delta \end{vmatrix}} = \frac{1}{\delta^4} + \frac{1}{\delta^2}, \quad (35)$$

$$A[\psi_3(\eta)] = \frac{\begin{vmatrix} \delta & 0 & 0 \\ 0 & \delta & \frac{1}{\delta} \\ 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} \delta & 0 & -1 \\ 0 & \delta & 1 \\ 1 & 1 & \delta \end{vmatrix}} = -\frac{1}{\delta^3}. \quad (36)$$

By applying the inverse AT on equations (34), (35) and (36), then we have

$$\psi_1(\eta) = A^{-1} \left[\frac{-1}{\delta^4} \right] = \frac{-\eta^2}{2}, \quad (37)$$

$$\psi_2(\eta) = A^{-1} \left[\frac{1}{\delta^4} + \frac{1}{\delta^2} \right] = 1 + \frac{\eta^2}{2}, \quad (38)$$

$$\psi_3(\eta) = A^{-1} \left[\frac{-1}{\delta^3} \right] = -\eta. \quad (39)$$

Equations (37), (38) and (39) give the solution of system (28) with the ICs (29). *Example 3.2.* Consider the following system of ODEs:

$$\left. \begin{aligned} \frac{d\psi_1}{d\eta} &= \psi_2(\eta) + \eta, \\ \frac{d\psi_2}{d\eta} &= \psi_1(\eta). \end{aligned} \right\} \quad (40)$$

With the conditions

$$\psi_1(0) = 1, \quad \psi_2(0) = 0. \quad (41)$$

The system (40) with the ICs (41) can be written:

$$\frac{d\psi}{d\eta} = B\psi(\eta) + \phi(\eta), \text{ with } \psi(0) = R, \quad (42)$$

where:

$$\frac{d\psi}{d\eta} = \begin{bmatrix} \frac{d\psi_1}{d\eta} \\ \frac{d\psi_2}{d\eta} \end{bmatrix}, H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \psi(\eta) = \begin{bmatrix} \psi_1(\eta) \\ \psi_2(\eta) \end{bmatrix},$$

$$\rho(\eta) = \begin{bmatrix} \eta \\ 0 \end{bmatrix}, \psi(0) = \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

By applying AT to the system (40), we get:

$$\left. \begin{aligned} A[\psi'_1(\eta)] + A[\psi_2(\eta)] &= A[\eta], \\ A[\psi_2(\eta)] - A[\psi_1(\eta)] &= 0. \end{aligned} \right\} \quad (43)$$

Running AT to the system (43) and using the ICs (41)

$$\left. \begin{aligned} \delta A[\psi_1(\eta)] - \frac{1}{\delta}\psi_1(0) + A[\psi_2(\eta)] &= \frac{1}{\delta^3}, \\ \delta A[\psi_2(\eta)] - \frac{1}{\delta}\psi_2(0) - A[\psi_1(\eta)] &= 0. \end{aligned} \right\} \quad (44)$$

Simplifying the system (44), we obtain

$$\left. \begin{aligned} \delta A[\psi_1(\eta)] + A[\psi_2(\eta)] &= \frac{1}{\delta} + \frac{1}{\delta^3}, \\ \delta A[\psi_2(\eta)] - A[\psi_1(\eta)] &= 0. \end{aligned} \right\} \quad (45)$$

Using Cramer's rule to solve $A[\psi_1(\eta)]$ and $A[\psi_2(\eta)]$ on the system (45):

$$A[\psi_1(\eta)] = \frac{\begin{vmatrix} \frac{1}{\delta} + \frac{1}{\delta^3} & 1 \\ 0 & \delta \end{vmatrix}}{\begin{vmatrix} \delta & 1 \\ -1 & \delta \end{vmatrix}} = \frac{1}{\delta^2}, \quad (46)$$

$$A[\psi_2(\eta)] = \frac{\begin{vmatrix} \delta & \frac{1}{\delta} + \frac{1}{\delta^3} \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} \delta & 1 \\ -1 & \delta \end{vmatrix}} = \frac{1}{\delta^3}. \quad (47)$$

By applying the inverse AT on equations (46) and (47), then we have

$$\psi_1(\eta) = A^{-1} \left[\frac{1}{\delta^2} \right] = 1, \quad (48)$$

$$\psi_2(\eta) = A^{-1} \left[\frac{1}{\delta^3} \right] = \eta. \quad (49)$$

Equations (48) and (49) give the solution of system (40) with the ICs (41).

Example 3.3. Consider the following system of ODEs:

$$\left. \begin{aligned} \frac{d\psi_1}{d\eta} &= \psi_2(\eta), \\ \frac{d\psi_2}{d\eta} &= \psi_1(\eta). \end{aligned} \right\} \quad (50)$$

with

$$\psi_1(0) = 1 \text{ and } \psi_2(0) = 1. \quad (51)$$

Sol. The system (50) with the ICs (51) can be written:

$$\frac{d\psi}{d\eta} = B\psi(\eta) + \phi(\eta), \text{ with } \psi(0) = R, \quad (52)$$

where:

$$\frac{d\psi}{d\eta} = \begin{bmatrix} \frac{d\psi_1}{d\eta} \\ \frac{d\psi_2}{d\eta} \end{bmatrix}, H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\psi(\eta) = \begin{bmatrix} \psi_1(\eta) \\ \psi_2(\eta) \end{bmatrix}, rho(\eta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \psi(0) = \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By applying AT to the system (50), we get:

$$\begin{cases} A[\psi'_1(\eta)] - A[\psi_2(\eta)] = 0, \\ A[\psi'_2(\eta)] - A[\psi_1(\eta)] = 0. \end{cases} \quad (53)$$

Running AT to the system (53) and using the ICs (51)

$$\begin{cases} \delta A[\psi_1(\eta)] - \frac{1}{\delta}\psi_1(0) - A[\psi_2(\eta)] = 0, \\ \delta A[\psi_2(\eta)] - \frac{1}{\delta}\psi_2(0) - A[\psi_1(\eta)] = 0. \end{cases} \quad (54)$$

Simplifying the system (54), we obtain

$$\begin{cases} \delta A[\psi_1(\eta)] - A[\psi_2(\eta)] = \frac{1}{\delta}, \\ \delta A[\psi_2(\eta)] - A[\psi_1(\eta)] = \frac{1}{\delta}. \end{cases} \quad (55)$$

Using Cramer's rule to solve $A\psi_1(\eta)$ and $A\psi_2(\eta)$ on the system (55)

$$A[\psi_1(\eta)] = \frac{\begin{vmatrix} \frac{1}{\delta} & -1 \\ \delta & -1 \end{vmatrix}}{\begin{vmatrix} \delta & -1 \\ -1 & \delta \end{vmatrix}} = \frac{1 + \frac{1}{\delta}}{\delta^2 - 1} = \frac{1}{\delta(\delta - 1)}, \quad (56)$$

$$A[\psi_2(\eta)] = \frac{\begin{vmatrix} \delta & \frac{1}{\delta} \\ -1 & \frac{1}{\delta} \end{vmatrix}}{\begin{vmatrix} \delta & -1 \\ -1 & \delta \end{vmatrix}} = \frac{1}{\delta(\delta - 1)}. \quad (57)$$

By applying the inverse AT on equations (56) and (57), then we have

$$\psi_1(\eta) = A^{-1} \left[\frac{1}{\delta(\delta - 1)} \right] = e^\eta, \quad (58)$$

$$\psi_2(\eta) = A^{-1} \left[\frac{1}{\delta(\delta - 1)} \right] = e^\eta. \quad (59)$$

Equations (58) and (59) give the solution of system (50) with the ICs (51).

IV. APPLICATION

This part of the research comprises an application of physical chemical problem for estimating the concentrations C_1 , C_2 and C_3 of the reactants Q , P and S of a chemical reaction of first order in batches given by the succeeding ODEs.

$$\left. \begin{cases} \frac{dC_1}{dt} = -\gamma_1 C_1, \\ \frac{dC_2}{dt} = \gamma_1 C_1 - \gamma_2 C_2, \\ \frac{dC_3}{dt} = \gamma_2 C_2, \end{cases} \right\} \quad (60)$$

with

$$C_1(0) = \varepsilon, C_2(0) = 0 \text{ and } C_3(0) = 0. \quad (61)$$

$$\left. \begin{cases} C_1 = C_1(t) = \text{Concentration of a chemical reactant } Q \\ \text{at time } t, \\ C_2 = C_2(t) = \text{Concentration of a chemical reactant } P \\ \text{at time } t, \\ C_3 = C_3(t) = \text{Concentration of a chemical reactant } S \\ \text{at time } t, \end{cases} \right\}$$

where $\gamma_1, \gamma_2 =$ rate constant > 0 .

$$\left. \begin{cases} C_1(0) = \varepsilon = \text{Concentration of a chemical reactant } Q \\ \text{at time } t = 0, \\ C_2(0) = 0 = \text{Concentration of a chemical reactant } P \\ \text{at time } t = 0, \\ C_3(0) = 0 = \text{Concentration of a chemical reactant } S \\ \text{at time } t = 0, \end{cases} \right\}$$

The system (60) with the ICs (61) can be expressed as:

$$\frac{dC}{dt} = BC + \phi(t), \text{ with } C(0) = R, \quad (62)$$

where

$$\frac{dC}{dt} = \begin{bmatrix} \frac{dC_1}{dt} \\ \frac{dC_2}{dt} \\ \frac{dC_3}{dt} \end{bmatrix}, B = \begin{bmatrix} -\gamma_1 & 0 & 0 \\ \gamma_1 & -\gamma_1 & 0 \\ 0 & \gamma_2 & 0 \end{bmatrix}, C(t) = \begin{bmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, C(0) = \begin{bmatrix} C_1(0) \\ C_2(0) \\ C_3(0) \end{bmatrix}, \text{ and } R = \begin{bmatrix} \varepsilon \\ 0 \\ 0 \end{bmatrix}.$$

By apply AT to the system (60), we get:

$$\begin{cases} A[C'_1(t)] + \gamma_1 A[C_1(t)] = 0, \\ A[C'_2(t)] - \gamma_1 A[C_1(t)] + \gamma_2 A[C_2(t)] = 0, \\ A[C'_3(t)] - \gamma_2 A[C_2(t)] = 0. \end{cases} \quad (63)$$

Running AT to the system (63) and using the ICs (61), we have

$$\left. \begin{cases} \delta A[C_1(t)] - \frac{1}{\delta}C_1(0) + \gamma_1 A[C_1(t)] = 0, \\ \delta A[C_2(t)] - \frac{1}{\delta}C_2(0) - \gamma_1 A[C_1(t)] + \gamma_2 A[C_2(t)] = 0, \\ \delta A[C_3(t)] - \frac{1}{\delta}C_3(0) - \gamma_2 A[C_2(t)] = 0. \end{cases} \right\} \quad (64)$$

Simplifying the system (64), we obtain

$$\left. \begin{aligned} (\delta + \gamma_1) A [C_1(t)] &= \frac{\varepsilon}{\delta}, \\ (\delta + \gamma_2) A [C_2(t)] - \gamma_1 A [C_1(t)] &= 0, \\ \delta A [C_3(t)] - \gamma_2 A [C_2(t)] &= 0. \end{aligned} \right\} \quad (65)$$

Using Cramer's rule to solve $A \{C_1(t)\}$, $A \{C_2(t)\}$ and $A C_3(t)$ on the system (65)

$$A[C_1(t)] = \frac{\begin{vmatrix} \delta + \gamma_1 & \frac{\varepsilon}{\delta} & 0 \\ -\gamma_1 & 0 & 0 \\ 0 & 0 & \delta \end{vmatrix}}{\begin{vmatrix} \delta + \gamma_1 & 0 & 0 \\ -\gamma_1 & \delta + \gamma_2 & 0 \\ 0 & -\gamma_2 & \delta \end{vmatrix}} \quad (66)$$

$$= \frac{\varepsilon(\delta + \gamma_2)}{\delta^3 + \delta^2\gamma_2 + \delta^3\gamma_1 + \delta\gamma_1\gamma_2},$$

$$A[C_2(t)] = \frac{\begin{vmatrix} \delta + \gamma_1 & \frac{\varepsilon}{\delta} & 0 \\ -\gamma_1 & 0 & 0 \\ 0 & 0 & \delta \end{vmatrix}}{\begin{vmatrix} \delta + \gamma_1 & 0 & 0 \\ -\gamma_1 & \delta + \gamma_2 & 0 \\ 0 & -\gamma_2 & \delta \end{vmatrix}} \quad (67)$$

$$= \frac{\varepsilon\gamma_1}{\delta^3 + \delta^2\gamma_2 + \delta^3\gamma_1 + \delta\gamma_1\gamma_2},$$

$$A[C_3(t)] = \frac{\begin{vmatrix} \delta + \gamma_1 & 0 & \frac{\varepsilon}{\delta} \\ -\gamma_1 & \delta + \gamma_2 & 0 \\ 0 & -\gamma_2 & 0 \end{vmatrix}}{\begin{vmatrix} \delta + \gamma_1 & 0 & 0 \\ -\gamma_1 & \delta + \gamma_2 & 0 \\ 0 & -\gamma_2 & \delta \end{vmatrix}} \quad (68)$$

$$= \frac{\varepsilon\gamma_1\gamma_2}{\delta} \left(\frac{1}{\delta^3 + \delta^2\gamma_2 + \delta^3\gamma_1 + \delta\gamma_1\gamma_2} \right).$$

By applying the inverse AT on equations (66), (67) and (68), then we have

$$C_1(t) = A^{-1} \left[\frac{\varepsilon(\delta + \gamma_2)}{\delta^3 + \delta^2\gamma_2 + \delta^3\gamma_1 + \delta\gamma_1\gamma_2} \right] = \varepsilon e^{-\gamma_1 t}, \quad (69)$$

$$C_2(t) = A^{-1} \left[\frac{\varepsilon\gamma_1}{\delta^3 + \delta^2\gamma_2 + \delta^3\gamma_1 + \delta\gamma_1\gamma_2} \right] \quad (70)$$

$$= \left(\frac{\varepsilon\gamma_1}{\gamma_2 - \gamma_1} \right) (e^{-\gamma_1 t} - e^{-\gamma_2 t}),$$

$$C_3(t) = A^{-1} \left[\frac{\varepsilon\gamma_1\gamma_2}{\delta} \left(\frac{1}{\delta^3 + \delta^2\gamma_2 + \delta^3\gamma_1 + \delta\gamma_1\gamma_2} \right) \right]$$

$$= \varepsilon \left(1 - \left(\frac{\gamma_2}{\gamma_2 - \gamma_1} \right) e^{-\gamma_1 t} + \left(\frac{\gamma_1}{\gamma_2 - \gamma_1} \right) e^{-\gamma_2 t} \right). \quad (71)$$

Equations (69), (70) and (71) give the solution of system (60) with the ICs (61). The values of concentrations C_1 , C_2 and C_3 are conformed to different values with time t and for different multiplications between, and they are determined and presented in Table II, and some graphical figures. Table II evinces that as time t augments from 0 to 6 sec.s, the concentration $C_1(t)$ of a chemical substance Q declines for

either combination of values of ε and γ_1 , namely

$$\left. \begin{aligned} \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.2 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.3 (sec^{-1}). \end{aligned} \right\}$$

Table II shows that as the rate constant γ_1 intensifies from 1 to $1.2sec^{-1}$, the concentration $C_1(t)$ of a chemical substance Q exhausts for the values of time t , spanning from 1 to 5 secs. Moreover, Table III delineates that for larger time, the value of $C_1(t)$ of a substance Q changes to 0 $\left(\frac{kg}{m^3} \right)$. The results presented in Table II are confirmed by Figure 1. Table III presents the concentration $C_2(t)$ of a chemical substance P , that decreases when t increases: 0 to 8 sec.s for all combinations of ε , γ_1 and γ_2 .

$$\left\{ \begin{aligned} \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.0 (sec^{-1}), \gamma_2 = 0.5 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.0 (sec^{-1}), \gamma_2 = 1.1 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.0 (sec^{-1}), \gamma_2 = 1.4 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.2 (sec^{-1}), \gamma_2 = 0.5 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.2 (sec^{-1}), \gamma_2 = 1.1 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.2 (sec^{-1}), \gamma_2 = 1.4 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.3 (sec^{-1}), \gamma_2 = 0.5 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.3 (sec^{-1}), \gamma_2 = 1.1 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.3 (sec^{-1}), \gamma_2 = 1.4 (sec^{-1}). \end{aligned} \right.$$

Table III shows that the value of rate constant γ_1 raises from 1 to $1.3 sec^{-1}$, the concentration $C_2(t)$ of a substance's chemical P has been raised, and it is diminished subsequently when the time t is increasing from 0 to 6 sec.s. Additionally, the table manifests that since the value of rate constant γ_2 goes from 0.6 to $1.4 sec^{-1}$, the value of a chemical substance's concentration P drops for all time t values.

Table IV illustrates the concentration $C_3(t)$ of a substance's chemical S increase when the time increases from 0 to 8 sec.s for some combinations of ε , γ_1 and γ_2 namely.

$$\left\{ \begin{aligned} \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.0 (sec^{-1}), \gamma_2 = 0.5 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.0 (sec^{-1}), \gamma_2 = 1.1 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.0 (sec^{-1}), \gamma_2 = 1.4 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.2 (sec^{-1}), \gamma_2 = 0.5 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.2 (sec^{-1}), \gamma_2 = 1.1 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.2 (sec^{-1}), \gamma_2 = 1.4 (sec^{-1}), \\ \varepsilon &= 1 \left(\frac{kg}{m^3} \right), \gamma_1 = 1.3 (sec^{-1}), \gamma_2 = 0.5 (sec^{-1}). \end{aligned} \right.$$

TABLE II
VALUES OF $C_1(t)$ AT TIME t FOR DIFFERENT COMBINATIONS OF CONSTANTS ϵ AND γ_1 .

t (sec)	$C_1(t)$ ($\frac{kg}{m^3}$) $\epsilon = 1 \text{ kg/m}^3,$ $\gamma_1 = 1 \text{ sec}^{-1}$	$\epsilon = 1 \text{ kg/m}^3,$ $\gamma_1 = 1.2 \text{ sec}^{-1}$	$\epsilon = 1 \text{ kg/m}^3,$ $\gamma_1 = 1.3 \text{ sec}^{-1}$
0	1.00	1.00	1.00
1.0	0.37	0.30	0.27
2.0	0.14	0.09	0.07
3.0	0.05	0.03	0.02
4.0	0.02	0.01	0.01
5.0	0.01	0.00	0.00
6.0	0.00	0.00	0.00
7.0	0.00	0.00	0.00
8.0	0.00	6.77	3.04

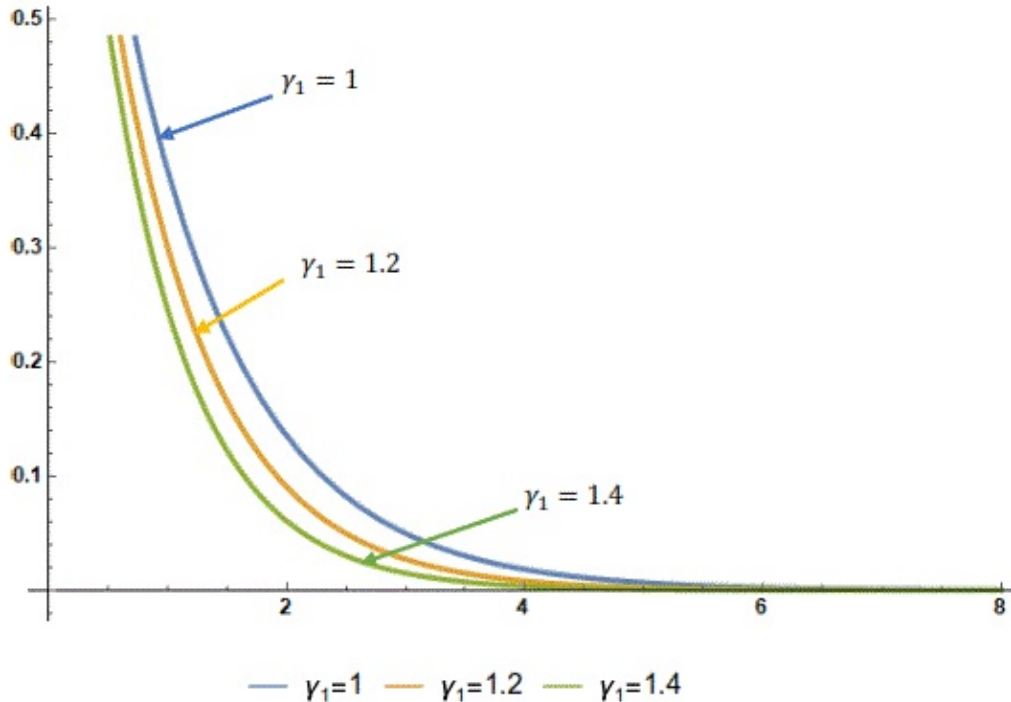


Fig. 1. Concentration $C_1(t)$ of a Q at time t multi values.

Table IV ensures that for high time t values, the concentration $C_3(t)$ of a substance's chemical S changes to $1 \frac{kg}{m^3}$. The graph presented in Figure 3 introduces the results discussed in Table IV.

V. CONCLUSION

In this research, we presented the AT, and we proved the basic properties and we showed the relation between it and some other integral transforms. Moreover, we solved some examples of systems of ODEs by AT. In the application section, the we looked at a problem from the field of physical chemistry to determine the concentration of chemical reactants of a chemical reaction in a chain and successfully solve them by the presented transform. The results of this research showed that AT solved the critical problem of focus determination of chemical reactants into a first-order chemical reaction respectively, at the time of the chemical reaction performance. The outcomes of the proposed research are beneficial to increase the production of things, by removing needless substances at the suitable time of the reaction. As a future work, AT can be utilized to solve the concentration of reactants in other chemical reactions.

REFERENCES

- [1] Y. Demirel, *Nonequilibrium thermodynamics: transport and rate processes in physical, chemical and biological systems*. Elsevier, 2007.
- [2] K. Velten, *Mathematical modeling and simulation: introduction for scientists and engineers*. John Wiley & Sons, 2009.
- [3] M. Ruth and B. Hannon, *Modeling dynamic biological systems*. Springer New York, 1997.
- [4] S. Elsayah, S. A. Pierce, S. H. Hamilton, H. V. Delden, D. Haase, A. Elmahdi, and A. J. Jakeman, "An overview of the system dynamics process for integrated modelling of socio-ecological systems: Lessons on good modelling practice from five case studies," *Environmental Modelling & Software*, vol. 93, pp. 127–145, 2017.
- [5] R. Khalil, S. Yasin, M. Hamad, A. Sharieh, and A. Al-Jaber, "Virtual reference values for str genetic loci assignment in forensic arenas: A jordanian-based study," *Jordan Journal of Biological Sciences*, vol. 1, no. 2, p. 273, 2008.
- [6] R. M. Khalil and R. Saadeh, "An algorithm of the prey and predator struggle to survive as a random walk simulation case study," in *International Conference on Mathematics and Computations*. Singapore: Springer Nature Singapore, May 2022, pp. 407–414.
- [7] A. Hazaymeh, R. Saadeh, R. Hatamleh, M. Alomari, and A. Qazza, "A perturbed milne's quadrature rule for n-times differentiable functions with lp-error estimates," *Axioms*, vol. 12, no. 9, 2023.
- [8] J. D. Farmer, M. Gallegati, C. Hommes, A. Kirman, P. Ormerod, S. Cincotti *et al.*, "A complex systems approach to constructing better models for managing financial markets and the economy," *The European Physical Journal Special Topics*, vol. 214, pp. 295–324, 2012.

TABLE III
CONCENTRATION $C_2(t)$ OF A CHEMICAL SUBSTANCE p AT TIME t FOR DIFFERENT COMBINATIONS OF VALUES ϵ , γ_1 , AND γ_2 .

t (sec)	$\epsilon = 1 \text{ kg/m}^3, \gamma_1 = 1 \text{ sec}^{-1}$			$\epsilon = 1 \text{ kg/m}^3, \gamma_1 = 1.2 \text{ sec}^{-1}$			$\epsilon = 1 \text{ kg/m}^3, \gamma_1 = 1.3 \text{ sec}^{-1}$		
	$\gamma_2 = 0.5$	$\gamma_2 = 1.1$	$\gamma_2 = 1.4$	$\gamma_2 = 0.5$	$\gamma_2 = 1.1$	$\gamma_2 = 1.4$	$\gamma_2 = 0.5$	$\gamma_2 = 1.1$	$\gamma_2 = 1.4$
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.0	0.48	0.35	0.29	0.52	0.38	0.33	0.54	0.39	0.34
2.0	0.47	0.25	0.17	0.48	0.24	0.18	0.48	0.24	0.18
3.0	0.35	0.13	0.08	0.34	0.11	0.07	0.33	0.11	0.07
4.0	0.23	0.06	0.03	0.22	0.05	0.03	0.21	0.04	0.02
5.0	0.15	0.03	0.01	0.14	0.02	0.01	0.13	0.02	0.01
6.0	0.09	0.01	0.00	0.08	0.01	0.00	0.08	0.01	0.00
7.0	0.06	0.00	0.00	0.05	0.00	0.00	0.05	0.00	0.00
8.0	0.04	0.00	0.00	0.03	0.00	0.00	0.03	0.00	0.00

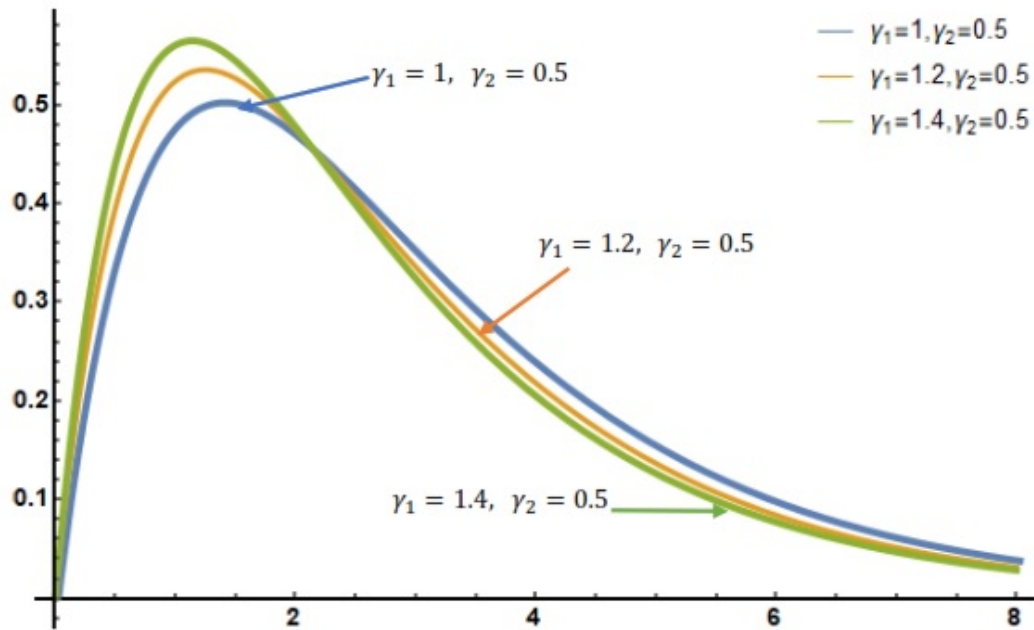


Fig. 2. Concentration $C_2(t)$ of a chemical substance Q at time t of different combinations of values ϵ , γ_1 and γ_2 .

[9] R. Saadeh, L. Hamdi, and A. Qazza, "Double laplace formable transform method for solving pdes," *Engineering Letters*, vol. 32, no. 3, pp. 579–587, 2024.

[10] L. J. Hettinger, A. Kirlik, Y. M. Goh, and P. Buckle, "Modelling and simulation of complex sociotechnical systems: envisioning and analysing work environments," *Ergonomics*, vol. 58, no. 4, pp. 600–614, 2015.

[11] S. M. Focardi and F. J. Fabozzi, *The mathematics of financial modeling and investment management*. John Wiley & Sons, 2004, vol. 138.

[12] K. Wolf, *Integral transforms in science and engineering*. Springer Science & Business Media, 2013, vol. 11.

[13] A. Qazza, "Solution of integral equations via laplace ARA transform," *European Journal of Pure and Applied Mathematics*, vol. 16, no. 2, p. 919–933, 2023.

[14] K. Rektorys, *Survey of applicable mathematics*. Springer, 2013, vol. 280.

[15] A. Qazza, "A new computation approach: ARA decomposition method," *WSEAS Transactions on Mathematics*, vol. 22, pp. 245–252, 2023.

[16] P. Deuffhard and F. Bornemann, *Scientific computing with ordinary differential equations*. Springer Science & Business Media, 2012, vol. 42.

[17] R. G. Rice, D. D. Do, and J. E. Maneval, *Applied mathematics and modeling for chemical engineers*. John Wiley & Sons, 2023.

[18] C. Chicone, *Ordinary differential equations with applications*. Springer Science & Business Media, 2006, vol. 34.

[19] I. Faragó, *Numerical methods for ordinary differential equations*. Budapest: TypoTech, 2014.

[20] J. C. Butcher, *Numerical methods for ordinary differential equations*. John Wiley & Sons, 2016.

[21] G. Rawitscher, V. dos Santos Filho, and T. C. Peixoto, *An Introductory Guide to Computational Methods for the Solution of Physics Problems*. Springer International Publishing, 2018.

[22] J. F. Epperson, *An introduction to numerical methods and analysis*. John Wiley & Sons, 2021.

[23] S. S. Ray, *Numerical analysis with algorithms and programming*. CRC Press, 2018.

[24] M. R. Spiegel, *Laplace transforms*. McGraw-Hill New York, 1965.

[25] R. K. Gupta, *Numerical methods: fundamentals and applications*. Cambridge University Press, 2019.

[26] H. P. Langtangen, "Solving nonlinear ode and pde problems," in *Center for Biomedical Computing, Simula Research Laboratory and Department of Informatics, University of Oslo*, 2016.

[27] A. A. Abed and W. T. Sotowski, "Finite element method algorithm for geotechnical applications based on runge-kutta scheme with automatic error control," *Computers and Geotechnics*, vol. 128, p. 103841, 2020.

[28] O. R. Al-Jayyousi, *Integral innovation: New worldviews*. Taylor & Francis, 2017.

[29] M. Allubbad, A. Qazza, and R. Saadeh, "Innovative approaches to linear voltaerra partial integro-differential equations: A laplace residual power series perspective," *IAENG International Journal of Computer Science*, vol. 51, no. 7, pp. 746–753, 2024.

[30] A. Brabazon and M. O'Neill, *Biologically inspired algorithms for financial modelling*. Springer Science & Business Media, 2006.

[31] M. Berir, R. Saadeh, M. A. Abdoon, A. Qazza, and D. Almutairi, "A fractional study for solving the sir model and chaotic system," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 2, pp. 212–217, 2024.

[32] K. Pilbeam, *Finance and financial markets*. Bloomsbury Publishing, 2018.

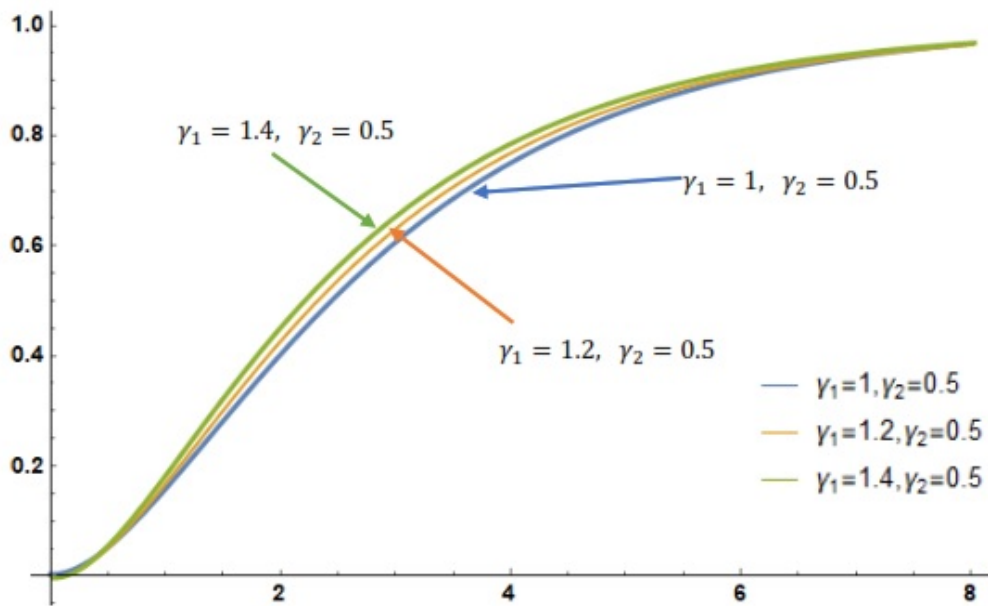


Fig. 3. Concentration $C_3(t)$ of a chemical substance Q at time t of different combinations of values ϵ , γ_1 and γ_2 .

TABLE IV
CONCENTRATION $C_3(t)$ OF A CHEMICAL SUBSTANCE p AT TIME t FOR DIFFERENT COMBINATIONS OF VALUES ϵ , γ_1 , AND γ_2 .

t (sec)	$\epsilon = 1 \text{ kg/m}^3, \gamma_1 = 1 \text{ sec}^{-1}$		$\epsilon = 1 \text{ kg/m}^3, \gamma_1 = 1.2 \text{ sec}^{-1}$			$\epsilon = 1 \text{ kg/m}^3, \gamma_1 = 1.3 \text{ sec}^{-1}$			
	$C_3(t) \left(\frac{\text{kg}}{\text{m}^3} \right)$		$\gamma_2 = 0.5$	$\gamma_2 = 1.1$	$\gamma_2 = 1.4$	$\gamma_2 = 0.5$	$\gamma_2 = 1.1$	$\gamma_2 = 1.4$	
	0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.0	0.15	0.28	0.33	0.18	0.32	0.37	0.18	0.34	0.39
2.0	0.40	0.62	0.68	0.43	0.67	0.73	0.45	0.69	0.75
3.0	0.60	0.82	0.86	0.64	0.86	0.90	0.65	0.87	0.91
4.0	0.75	0.92	0.95	0.77	0.94	0.96	0.78	0.95	0.97
5.0	0.84	0.97	0.98	0.86	0.98	0.99	0.87	0.98	0.99
6.0	0.90	0.99	0.99	0.92	0.99	1.00	0.92	0.99	1.00
7.0	0.94	0.99	1.00	0.95	1.00	1.00	0.95	1.00	1.00
8.0	0.96	1.00	1.00	0.97	1.00	1.00	0.97	1.00	1.00

[33] W. Wasylkiwskyj, *The Z-Transform and Discrete Signals*. New York, NY: Springer New York, 2013, pp. 311–336.

[34] R. Khalil and A. Al-Jumaily, “Hybridization of local search optimization and support vector machine algorithms for classification problems enhancement,” in *Intelligent Computing Paradigm and Cutting-edge Technologies: Proceedings of the First International Conference on Innovative Computing and Cutting-edge Technologies (ICICCT 2019)*. Istanbul, Turkey: Springer International Publishing, Oct. 2020, pp. 209–217.

[35] O. Ala’yed, R. Saadeh, and A. Qazza, “Numerical solution for the system of lane-emen type equations using cubic b-spline method arising in engineering,” *AIMS Mathematics*, vol. 8, no. 6, p. 14747 – 14766, 2023.

[36] K. Chandan, R. Saadeh, A. Qazza, K. Karthik, R. Kumar, R. Kumar, U. Khan, M. A. M. Abdou, W. Ojok, and R. Kumar, “Predicting the thermal distribution in a convective wavy fin using a novel training physics-informed neural network method,” *Scientific Reports*, vol. 14, no. 1, 2024.

[37] A. Alzahrani, R. Saadeh, M. Abdoon, M. Elbadri, M. Berir, and A. Qazza, “Effective methods for numerical analysis of the simplest chaotic circuit model with atangana–baleanu caputo fractional derivative,” *Journal of Engineering Mathematics*, vol. 144, no. 1, 2024.

[38] R. Saadeh, O. Ala’yed, and A. Qazza, “Analytical solution of coupled hirota–satsuma and kdv equations,” *Fractal and Fractional*, vol. 6, no. 12, 2022.

[39] R. Saadeh, M. Abu-Ghuwaleh, A. Qazza, and E. Kuffi, “A fundamental criteria to establish general formulas of integrals,” *Journal of Applied Mathematics*, vol. 2022, 2022.

[40] A. E. Hamza, A. K. Sedeeg, R. Saadeh, A. Qazza, and R. Khalil, “A new approach in solving regular and singular conformable fractional coupled burger’s equations,” *WSEAS Transactions on Mathematics*, vol. 22, pp. 298–314, 2023.

[41] K. S. Aboodh, “The new integral transform ‘aboodh transform,’” *Global Journal of Pure and Applied Mathematics*, vol. 9, no. 1, pp. 35–43, 2013.

[42] S. A. Altaie, N. Anakira, A. Jameel, O. Ababneh, A. Qazza, and A. K. Alomari, “Homotopy analysis method analytical scheme for developing a solution to partial differential equations in fuzzy environment,” *Fractal and Fractional*, vol. 6, no. 8, p. 419, 2022.