Some Theorems on Fixed Points in N-Cone Metric Spaces with Certain Contractive Conditions

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Abstract—**This work presents the idea of a symmetric N-cone metric space and explore the convergence properties of sequences within this framework. Furthermore, we show fixed-point theorems for maps that satisfy certain contractive conditions and present examples to illustrate our point. Our findings complement, expand, and consolidate numerous established conclusions in the literature.**

Index Terms—**cone metric space, N-cone metric, Cauchy sequence, fixed point theorems.**

I. Introduction

NUMEROUS mathematical problems have solutions that imply the existence of a fixed point, assuming that the function in question is appropriate. Thus, the existence of a fixed point is crucial in many branches of mathematics and other sciences. Results at fixed points are necessary for the existence of solutions in maps. A formulation for locating the fixed points of a nonlinear mapping is frequently employed in applied and analytical mathematics to resolve issues involving nonlinear functional equations. Fixed point theorems are really important in verifying the existence and uniqueness of solutions for numerous mathematical models. These models include variational inequalities, partial differential equations, economic theory, steady state temperature distribution, and biological research. They depict occurrences in several domains, including fluid dynamics, economic principles, financial evaluation, disease outbreaks, and biological investigations. In their work, Huang and Zhang [5] created a cone metric space and established theorems about fixed points in contractive mappings. Rezapour and Halbarani [4] eliminated the necessity for normality in cone metric space. Subsequently, other articles and references related to cone metric space started to emerge (refer to [1], [2], [4], [6], [9]–[12], [17]). It should be noted that the findings presented in this work were not obtained by using the normalcy feature of the cone. Our findings have several implications, one of which is the generalization of similar findings in previous research (see [8], [13]–[16]) and the citations therein. This work may be seen as laying the theoretical groundwork for future applications to real-world situations, driven by the importance of fixed-point theory and its practical applications, especially when combined with multi-dimensional theory (refer to references [18]–[22]) and the citations therein. This work presents convergence criteria for sequences in N-cone metric spaces and establishes fixed-point theorems for mappings that meet certain contractive type constraints.

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II. Preliminaries

In this section, we will examine and discuss the essential terminology and results that are required for the next sections. Consider a real Banach space E and a subset P of E. Then, a cone is defined as P if and only if

1) The set P is nonempty, closed, and not equal to the singleton set $\{0\}$.;
2) For every x and y belonging to the set P, and $a, b \geq 0$, the expression $ax + by$ also belongs to the set P.;
3) $P \cap (-P) = \{0\}$.

Within E, one may construct a partial ordering $\leq$ for a cone P. In this ordering, $x \leq y$ if and only if $y - x$ is an element of P. We use $x < y$ to represent the relationship between x and y where x is less than y, but they are not equal. Additionally, $x << y$ is used to indicate that the difference between y and x is within $Int.\ P$.

A cone is considered normal if there exists a positive number M such that for any x and y in E, the inequality $\parallel x \parallel \leq M \parallel y \parallel$ holds. The smallest positive number that satisfies the given condition is known as the normal constant of P.

A cone P is considered regular if any increasing sequence that has an upper bound converges. In other words, if we have a sequence $\{x_n\}_{n\geq 1}$ where $u_1 \leq u_2 \leq \cdots \leq v$ for some $v \in E$, there exists an element $u \in E$ such that $\lim_{n\to\infty} \parallel u_n - u \parallel = 0$. Similarly, the cone P is considered regular if and only if any decreasing sequence that has a lower bound will eventually converge.

Lemma 1. *[8] A regular cone is a normal cone.*

Definition 2. *[7] Assume that X is a non-empty set. Then the function $N : X^3 \to E$ defines a N-cone metric space on X if and only if for every $u, v, w, b \in X$, it meets the following conditions.*

1) $N(u, v, w) \geq 0$;
2) $N(u, v, w) = 0$ *iff* $u = v = w$;
3) $N(u, v, w) \leq N(u, u, b) + N(v, v, b) + N(w, w, b)$.

Then the function N defines a N-cone metric, and the set (X, N) is a N-cone metric space.

Proposition 3. *[7] Consider the N-cone metric space (X, N). Then $N(u, u, v) = N(v, v, u)$ holds for every $u, v, w \in X$.*

Definition 4. *[7] Consider the N-cone metric space (X, N) and let a sequence $\{u_n\}$ in X and a point $u \in X$. If $\{u_n\}$ is convergent for every $n > M$, then $N(u_n, u_n, u) \leq a$ holds for any $a \in E$ where $0 << a$. The sequence $\{u_n\}$ converges to $u \in X$. We write by $u_n \to u$ as $n \to \infty$.*

Lemma 5. *[7] Consider the N-cone metric space (X, N) and let $P \subseteq E$ be a normal cone with a normal constant M. Consider a sequence $\{u_n\}$ in X. It follows that $u = v$ if and only if $\{u_n\}$ converges to both u and v, and there is only one unique limit of $\{u_n\}$, if it exists.*

Definition 6. *[7] Assume (X, N) is a N-cone metric space with $\{u_n\} \in X$. If for every $b \in E$ with $0 << b$ there exists an M such that for $m, n > M$, $N(u_n, u_n, u_m) << b$, then $\{u_n\}$ is called a Cauchy sequence in X.*

Definition 7. *[7] Assume (X, N) is a N-cone metric space. If every Cauchy sequence $\{u_n\} \in X$ converges to $u \in X$, then X is defined as a complete N-cone metric space.*

Lemma 8. *[7] Assume (X, N) is a N-cone metric space with $\{u_n\} \in X$. If $\{u_n\}$ converges to u, then $\{u_n\}$ is a Cauchy sequence.*

Definition 9. *[7] Consider the N-cone metric spaces (X, N) and $(X^{'}, N^{'})$. A function $T : X \to X^{'}$ is continuous at a point $u \in X$ if and only if it is sequentially continuous at u. In other words, If a sequence in $u_n \in X$ converges to u, then $\{Tu_n\}$ also converges to Tu.*

Lemma 10. *[7] Assume (X, N) be a N-cone metric space, and a normal cone $P \subseteq E$ with the normal constant M. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in X such that $u_n \to u, v_n \to v$ as $n \to \infty$. Then $N(u_n, u_n, v_n) \to N(u, u, v)$ as $n \to \infty$.*

Remark 11. *[7] If the sequence $u_n \in X$ converges to $u \in X$ where (X, N) is a N-cone metric space, then every subsequence of $\{u_n\}$ also converges to u.*

Proposition 12. *[7] Assume (X, N) is a N-cone metric space, with $P \subseteq E$ a real Banach space cone. Then $v \ll w$ and $u \ll v$ implies that $u \ll w$.*

Definition 13. *[3] Consider the N-cone metric space (X, N). Then, the map $g : X \to X$ is considered a contractive mapping if there exists a constant $0 \leq m < 1$ such that the inequality*

$$N(\ g(u),\ \ g(v),\ \ g(v)) \leq m\, N(u, u, v)$$

holds for every u, v in X.

Lemma 14. *[7] Consider N-cone metric space (X, N). Let $P \subseteq E$ be a cone in a real Banach space. Assume that k_1, k_2, k_3, k_4, k are all positive. If the sequences u_n, v_n, w_n, and q_n converge to u, v, w, and q respectively in the space X, then*

$$\begin{aligned} kb &\leq k_1 N(u_n, u_n, u) + k_2 N(v_n, v_n, v) \\ &\quad + k_3 N(w_n, w_n, w) + k_4 N(q_n, q_n, q) \\ \Rightarrow b &= 0 \end{aligned}$$

III. N-Cone Metric Spaces

In this section, we define a symmetric N-cone metric space and establish the convergence properties of sequences.

Definition 15. *A N-cone metric space (X, N) is considered symmetric if*

$$N(u,\ v,\ v) = N(\ v,\ u,\ u)$$

for all u, v in X.

Example 16. *Take (X, d) as a cone metric space. Define*

$$N : X^3 \to E$$

by

$$N(u, v, w) = d(u,\ v) + d(v,\ w) + d(w,\ u)$$

for every $u, v, w \in X$. Then X is a symmetric N-cone metric space.

Example 17. *Let $X = \{c, d\}, E = R^3$ and*

$$P = \{(u, v, w) \in E : u, v, w \geq 0\}.$$

Define $N : X^3 \to E$ by

$$N(c,\ c,\ c) = (0,\ 0,\ 0) = N(d,\ d,\ d),$$

$$N(c,\ d,\ d) = (0, 1, 1) = N(d,\ c, d) = N(d,\ d, c),$$

$$N(d,\ c,\ c) = (0, 1, 0) = N(c,\ d,\ c) = N(c,\ c,\ d),$$

Then X is non-symmetric N-cone metric space.

Definition 18. *Let (X, N) be a N-cone metric space, define*

$$d_N : X \times X \to E$$

by

$$d_N(u,\ v) = N(u,\ v, v) + N(v,\ u, u)$$

for all $u, v \in X$. Then (X, d_N) is N-cone metric space. If X is a symmetric N-cone metric space, then

$$d_N(u,\ v) = 2N(u,\ v,\ v),$$

for every $u, v \in X$.

Lemma 19. *Consider the N-cone metric space (X, N) and let $\{u_m, v_n\}$ and $\{w_l\}$ be sequences in X such that $u_m \to u$, $v_n \to v$ and $w_l \to w$, then $N(u_m, v_n, w_l) \to N(u, v, w)$ as m, n, and l tends to infinity.*

Proof: We know that $\{u_m, v_n\}$ and $\{w_l\}$ converge to u, v, w, respectively. For any $c \in E$ such that $0 << c$, there exists M such that for every m, n, and l greater than M. Then we obtain

$$N(u_m, u, u) << \frac{c}{3}, \quad N(v_n, v, v) << \frac{c}{3}$$

and

$$N(w_l, w, w) << \frac{c}{3}.$$

Since,

$$N(u_m, v_n, w_l) \leq N(u_m, u,\ u) +\ N(\ u,\ v_n, w_l),$$

$$N(u, v_n,\ w_l) \leq N(v_n,\ v,\ v) +\ N(\ v,\ u,\ w_l).$$

and

$$N(v, u, w_l) \leq N(w_l, w, w) + N(w, v, u).$$

So,

$$N(u_m, v_n, w_l) \leq N(u_m, u, u) + N(v_n, v, v) + N(w_l, w, w) + N(w, v, u).$$

Then

$$N(u_m, v_n, w_l) - N(u, v, w) \leq N(u_m, u, u) + N(v_n, v, v) + N(w_l, w, w) << \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

Thus

$$N(u_m, v_n, w_l) - N(u, v, w) << c.$$

Similarly,

$$N(u, v, w) - N(u_m, v_n, w_l) << c.$$

So, for any $k \geq l$, we have

$$N(u_m, v_n, w_l) - N(u, v, w) << \frac{c}{k},$$

and

$$N(u, v, w) - N(u_m, v_n, w_l) << \frac{c}{k}.$$

It clearly shows that

$$\frac{c}{k} - (N(u, v, w) - N(u_m, v_n, w_l)$$

and

$$\frac{c}{k} + N(u_m, v_n, z_l) - N(u, v, w)$$

belongs to P. Given that P is closed and $\frac{c}{k} \to 0$ as $k \to \infty$, then

$$\lim_{m,n,l \to \infty} N(u_m, v_n, w_l) - N(u, v, w)$$

and

$$N(u, v, w) - \lim_{m,n,l \to \infty} N(u_m, v_n, w_l) \in P.$$

This gives that

$$\lim_{m,n,l \to \infty} N(u_m, v_n, w_l) = N(u, v, w)$$

■

Lemma 20. *Consider a sequence $\{u_n\}$ in the N-cone metric space (X, N), where $u \in X$. If $\{u_n\}$ converges to both u and v, then u must be equal to v.*

Proof: For any $c \in E$, where $0 << c$, there exists a M such that for any $m, n > M$,

$$N(u_m, u_n, u) << \frac{c}{3}$$

and

$$N(u_m, u_n, v) << \frac{c}{3}.$$

Then

$$N(u, u, v) \leq N(u_m, u_n, \ v) + \ N(\ u_m, u_n, u),$$

and

$$N(u_n, u, v) \leq N(u_m, u_n, v) + N_{u_m, u_n, u}.$$

Hence

$$N(u, u, v) \leq N(u_n, u_n, u) + N(u_m, u_n, v) + N(u_m, u_n, u) << \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

Thus, $N(u, u, v) << \frac{c}{m}$ for all $m \geq 1$. So,

$$\frac{c}{m} - N(u, u, v) \in P,$$

For every $m \geq 1$. P is closed and since $\frac{c}{m} \to 0$ as $m \to \infty$, therefore $-N(u, u, v) \in P$. but

$$N(u, u, v) \in P \Rightarrow N(u, u, v) = 0.$$

Thus $u = v$. ■

Lemma 21. *Assume $\{u_n\}$ is a sequence in a N-cone metric space (X, N), with $u \in X$. If $\{u_n\}$ converges to u and $m, n \to \infty$, then $N(u_m, u_n, u) \to \infty$.*

Proof: Let $\{u_n\}$ converges to u for $u \in X$. There exists M such that for $m, n > M$, the following condition holds:

$$N(u_n, u_m, u) << c.$$

Hence, we can say that for any $K \geq 1$,

$$N(u_n, u_m, u) << \frac{c}{k},$$

Therefore, Clearly,

$$\frac{c}{k} - N(u_n, u_m, u) \in P,$$

thus

$$- \lim_{n,m \to \infty} N(u_n, u_m, u) \in P.$$

But

$$\lim_{n,m \to \infty} N(u_n, u_m, u) \in P.$$

Because of this, it provides

$$\lim_{n,m \to \infty} N(u_n, u_m, u) = 0.$$

■

Lemma 22. *Consider a sequence $\{u_n\}$ in N-cone metric space (X, N). Let u be an element of X. Then a sequence $\{u_n\}$ is said to be a Cauchy sequence if it converges to u.*

Proof: Let's say that $c \in E$ and $0 << c$. If $u_n \to u$, then there is a number M, such that for all $m, n, l > M$, we have

$$N(u_m, u_n, u) << \frac{c}{3}, \quad N(u_n, u_l, u) << \frac{c}{3}$$

and

$$N(u_m, u_l, u) << \frac{c}{3}.$$

Then

$$N(u_m, u_n, u_l) \leq N(u_m, u_n, u) + N(u_n, u_l, u) + N(u_m, u_l, u) << c$$

Thus, $\{u_n\}$ is a Cauchy sequence. ■

Lemma 23. *Consider a sequence $\{u_n\}$ in the N-cone metric space (X, N). Let $\{u_n\}$ be a Cauchy sequence in X, then*

$$N(u_m, u_n, u_l) \to 0$$

as m, n, l tends to ∞.

Proof: Let $\{u_n\}$ denote a Cauchy sequence in X. For any element c in E such that $0 \ll c$, there exists M such that for all values $m, n, l > M$,

$$N(u_m, u_n, u_l) << c.$$

So, for any $k \geq 1$, we have

$$N(u_n, u_m, u_l) << \frac{c}{k}.$$

clearly,

$$\frac{c}{k} - N(u_n, u_m,\ u_l) \in P,$$

thus

$$-\lim_{n,m,l \to \infty} N(\ u_n, u_m,\ u_l) \in P.$$

But

$$\lim_{n,m,l \to \infty} N(u_n, u_m, u_l) \in P.$$

Thus

$$\lim_{n,m,l \to \infty} N(u_n, u_m,\ u_l) =\ 0$$

■

IV. Theorems on Fixed Points

Here, we shall establish fixed-point theorems for symmetric N-cone metric spaces.

Theorem 24. *Consider a mapping $T : X \to X$ that fulfills one of the following conditions, given that (X, N) is a complete symmetric N-cone metric space:*

$$N(\ Tu,\ Tv,\ Tw) \leq d\, N(w, Tw, Tw) + c\, N(v, Tv, Tv) + b\, N(u, Tu, Tv) + a\, N(u, v, w) \tag{1}$$

Or

$$N(Tu, Tv, Tw) \leq d\, N(w, w, Tw) + c\, N(v, v, Tv) + b\, N(u, Tu, u) + a\, N(u, v, w) \tag{2}$$

where $0 \leq d + c + b + a < 1$, for every u, v, w in X. Then there is a unique fixed point for T.

Proof: Assume that T holds condition (1); therefore, for any $u, v \in X$,

$$N(\ Tu,\ Tv, Tv) \leq (d + c)\ N(v,\ Tv,\ Tv) + a\ N(u, v, v) + b\ N(u, Tu, Tu) \tag{3}$$

and

$$N(Tv,\ Tu,\ Tu) \leq (d + c)\ N(u, Tu, Tu) + a\ N(v,\ u,\ u) + bN(\ v,\ Tv,\ Tv) \tag{4}$$

Given that (X, N) is a symmetric N-cone metric space, we may deduce from (3) and (4) that

$$\begin{aligned} d_N(Tu, Tv) &\leq a d_N(u, v) + \frac{d + c + b}{2} d_N(u, Tu) \\ &+ \frac{d + c + b}{2} d_N(v, Tv) \\ &= \alpha d_N(u, v) + \beta d_N(u, Tu) \\ &+ \gamma d_N(v, Tv) \end{aligned} \tag{5}$$

where $\beta = \gamma = \frac{d+c+b}{2}, \alpha = a$, for all u, v in X. It is evident that $\alpha + \beta + \gamma < 1$. Pick any u in X. Consider the sequence $\{T^n u\}$ for $u \in X$. In (5), we can obtain the following by substituting u with $T^n u$ and v with $T^{n-1} u$.

$$\begin{aligned} d_N(T^{n+1}u, T^n u) \leq \alpha d_N(T^n u, T^{n-1} u) \\ + \beta d_N(T^n u, T^{n+1} u) \\ + \gamma d_N(T^{n-1} u, T^n u) \end{aligned}$$

Additionally, it gives that

$$d_N(T^{n+1}u, T^n u) \leq p d_N(T^n u, T^{n-1} u)$$

where $p = \frac{\alpha+\beta}{1-\beta} < 1$. Therefore, it is evident that

$$d_N(T^{n+1}u, T^n u) \leq \frac{p^n}{1-p} d_N(Tu, u)$$

Let $0 << c$ be given. Then we can conclude that $\frac{p^n}{1-p} d_N(Tu, u) << c$. So we have $d_N(T^m u, T^n u) << c$, for $m > n$. Therefore $\{T^n u\}$ is a Cauchy sequence and hence $T^n u \to w$.Now we shall show that $Tw = w$. Now we shall show that $T^{n+1}u \to Tw$. Take $u = T^n, v = w$ in (5),

$$\begin{aligned} d_N(T^{n+1}u, Tw) &\leq \alpha d_N(T^n u, w) \\ &+ \beta d_N(T^n u, T^{n+1} u) + \gamma d_N(Tw, w) \\ &\leq \alpha d_N(T^n u, w) + \beta d_N(T^n u, T^{n+1} u) \\ &+ \gamma (d_N(T^{n+1} w, Tw) + d_N(Tw, w)) \\ &\leq \alpha d_N(T^n u, w) + \beta p^n d_N(Tu, u) \\ &+ \gamma (d_N(T^{n+1} v, Tw) + d_N(T^{n+1} u, w)) \end{aligned}$$

Thus

$$\begin{aligned} d_N(T^{n+1}u, Tw) \leq \frac{1}{1-\gamma} (\alpha d_N(T^n u, w) \\ + \beta p^n d_N(T^n u, u) + \gamma d_N(T^{n+1} u, w)) << c \end{aligned}$$

for every $c \in E$. This demonstrates that $T^{n+1}u \to Tw$ as $n \to \infty$. Now,

$$\begin{aligned} d_N(T^{n+1}u, Tw) &\leq d_N(T^{n+1}u, w) \\ &+ d_N(T^{n+1}u, Tw) \\ &<< \frac{c}{2} + \frac{c}{2} = c \end{aligned}$$

at any time $n > N$. Thus $d_N(Tw, w) << \frac{c}{m}$, for $m \geq 1$. So, for all $m \geq 1$, $\frac{c}{m} - d_N(Tw, w) \in P$. P is closed and again since $\frac{c}{m} \to 0$ as $m \to \infty$, therefore, $d_N(Tw, w)$ belongs to P provides $d_N(Tw, w) = 0$, thus $Tw = w$. ■

Remark 25. *By applying the results from the previous theorem to conditions (3) and (4), we may obtain the following if X is not a symmetric N-cone metric space:*

$$\begin{aligned} d_N(Tu, Tv) \leq a d_N(\ u,\ v) \\ + \frac{2(d + c + b)}{3} d_N(\ u,\ T\ u) \\ + \frac{2(d +\ c +\ b)}{3} d_N(v, T\ v) \end{aligned}$$

for all $u, v \in X$. Here, $0 \leq a + \frac{2(d+c+b)}{3} + \frac{2(d+c+b)}{3}$ which is at least 1. Therefore, the above theorem provides no information.

Theorem 26. *Let a mapping $T : X \to X$ that satisfies either condition (1) or condition (2), given that (X, N) is a complete N-cone metric space. Therefore, T possesses a unique fixed point.*

Proof: Let $u_0 \in X$. Define a sequence $\{u_n\}$ by

$$u_n = T^n(u_0).$$

As a result of (1), we get

$$\begin{aligned} N(u_n, u_{n+1}, u_{n+1}) \leq\ & aN(u_{n-1}, u_n, u_n) \\ & + bN(u_{n-1}, u_n, u_n) \\ & + (c+d)N(u_n, u_{n+1}, u_{n+1}) \end{aligned}$$

It suggests that

$$N(u_n, u_{n+1}, u_{n+1}) \leq qN(u_{n-1}, u_n, u_n),$$

where $q = \frac{a+b}{1-c-d}$. It is clear that $0 \leq q < 1$. Proceed using this approach to obtain

$$N(u_n, u_{n+1}, u_{n+1}) \leq q^n N(u_0, u_1, u_2).$$

Moreover for all $n, m \in N$ with $m > n$, we have

$$\begin{aligned} N(u_n, u_m, u_m) \leq\ & N(u_n, u_{n+1}, u_{n+1}) \\ & + N(u_{n+1}, u_{n+2}, u_{n+2}) \\ & + \cdots + N(u_{m-1}, u_m, u_m) \\ \leq\ & (q^n + q^{n+1} + \cdots + q^{m-1}) \\ & N(u_0, u_1, u_1) \\ \leq\ & \frac{q^n}{1-q} N(u_0, u_1, u_1) \end{aligned}$$

Assume $0 << c$. Choose $\delta > 0$, where $N_\delta(0) = \{v \in E :\| v \|< \delta\}$ such that $c + N_\delta(0) \subseteq P$. Choose a positive integer, N_1 such that $\frac{q^n}{1-q} N(u_0, u_1, u_1) \in N_\delta(0)$, for all $m > N_1$. Then, $\frac{q^n}{1-q} N(u_0, u_1, u_1) << c$, for every $m \geq N_1$. Thus, for all $m > n$, $N(u_n, u_m, u_m) << c$. Thus $\{u_n\}$ is a Cauchy sequence. In the set X, there is an element x to which the sequence $\{u_n\}$ converges. Now, starting with (1)

$$\begin{aligned} N(u_n, Tx, Tx) \leq\ & a\ N(u_{n-1},\ x,\ x) \\ & + bN(u_{n-1}, u_n,\ u_n) + (d+c)\ N(x, Tx, Tx) \end{aligned}$$

As $n \to \infty$, we obtain

$$N(x, Tx, Tx) \leq (c+d)N(x, Tx, Tx)$$

it indicates $T(x) = x$. So as to establish uniqueness, we are going to presume that $x \neq y = T(y)$, then

$$\begin{aligned} N(x, y, y) \leq\ & aN(x, y, y) + bN(x, Tx, Tx) \\ & + (c+d)N(y, Ty, Ty) \\ & + aN(x, y, y) \\ \Rightarrow\ & x = y \end{aligned}$$

Thus T has a unique fixed point. ■

Theorem 27. *Let a mapping $T : X \to X$ that satisfies one of conditions where the metric space (X, N) is complete N-cones space:*

$$\begin{aligned} N(Tu, Tv, Tv) \leq\ & a\{N(u, Tv, Tv) \\ & + N(v, Tu, Tu)\} \end{aligned} \tag{6}$$

or

$$\begin{aligned} N(Tu, Tv, Tv) \leq\ & a\{N(v, v, Tu) \\ & + N(u, u, Tv)\} \end{aligned} \tag{7}$$

for all $u, v \in X$ where $0 \leq a \leq \frac{1}{2}$, then T possesses a unique fixed point.

Proof: Assuming T holds condition (6), it follows that for any u and v in X,

$$\begin{aligned} N(Tu,\ Tv,\ Tv) \leq\ & a\{N(u,\ Tv,\ Tv) \\ & + N(\ v,\ Tu,\ Tu)\} \end{aligned}$$

and

$$\begin{aligned} N(Tv, Tu, Tu) \leq\ & a\{N(v, Tu, Tu) \\ & + N(u, Tv, Tv)\}. \end{aligned}$$

When considering a symmetric N-cone metric space, the inequalities mentioned above yield the following result

$$d_N(Tu, Tv) \leq a\{d_N(u, Tv) + d_N(v, Tu)\},$$

for every u, v in X. Since $0 \leq a < \frac{1}{2}$, hence, the outcome is as follows. We get by combining (6) and (7) when $(X, N$ is not a symmetric N-cone metric space:

$$\begin{aligned} & d_N(\ Tu,\ Tv) = N(\ Tu,\ Tv,\ Tv) + N(\ Tv,\ Tu,\ Tu) \\ & \leq 2a\{N(v, Tu, Tu) + N(u, Tv, Tv)\} \\ & \leq \frac{4a}{3}\{d_N(\ v, Tu) + d_N(u,\ Tv)\} \end{aligned}$$

for all u, v in X. In this case, the N-cone metric space provides no information if the contractivity factor $\frac{4a}{3}$ may not be less than 1. Therefore, suppose we have an initial value u_0 in the set X, and let's define a sequence, $\{u_n\}$ by $u_n = T^n u_0$. So by (6)

$$\begin{aligned} N(u_n, u_{n+1}, u_{n+1}) \leq\ & a\{N(u_n,\ u_n,\ u_n) \\ & + N(u_{n-1},\ u_n,\ u_n)\} \\ =\ & aN(u_{n-1},\ u_{n+1},\ u_{n+1}) \end{aligned}$$

But

$$\begin{aligned} N(u_n, u_{n+1}, u_{n+1}) \leq\ & N(u_{n-1}, u_n, u_n) \\ +\ & N(u_n, u_{n+1}, u_{n+1}) \end{aligned}$$

Thus we have

$$N(u_n, u_{n+1}, u_{n+1}) \leq kN(u_{n-1}, u_n, u_n),$$

$0 \leq k < 1$, where $k = \frac{a}{1-a}$. Continuing with the previous procedure, we obtain: Thus we have

$$N(u_n, u_{n+1}, u_{n+1}) \leq k^n N(u_0, u_1, u_1)$$

Thus $\{u_n\}$ is a Cauchy sequence, then $\{u_n\}$ converges to x, hence there is a $x \in X$. It is time to show that $Tx = x$. From (6)

$$\begin{aligned} N(u_n, Tx, Tx) \leq\ & a\{N(u_{n-1},\ Tx,\ Tx) \\ & + N(x,\ u_n,\ u_n)\}. \end{aligned}$$

As $n \to \infty$, limiting implies that

$$N(x, Tx, Tx) \leq aN(x, Tx, Tx)$$

Thus $Tx = x$. Suppose that $x \neq y = Ty$, then

$$N(x,\ y,\ y) \leq a\{N(y,\ x,\ x) + N(\ x, y,\ y)\}.$$

so

$$N(x,\ y,\ y) \leq k\ N(\ y, x,\ x).$$

again, we have

$$N(x, y, y) \leq k^2 N(x, y, y).$$

which implies that $x = y$. ■

Example 28. *Let* $E = R^3$

$$P = \{(u, v, w) \in R^3 : u, v, w \geq 0\}$$

and

$$X = \{(u, 0, 0) \in R^3 : 0 \leq u \leq 1\}$$
$$\cup\{(0, u, 0) \in R^3 : 0 \leq u \leq 1\}$$
$$\cup\{(0, 0, u) \in R^3 : 0 \leq u \leq 1\}.$$

Define a mapping

$$N : X^3 \to E$$

by

$$\begin{aligned} &N((u, 0, 0), (v, 0, 0), (w, 0, 0)) \\ &= (\frac{4}{3}(\mid v - w \mid) + \mid u - v \mid, \\ &\mid v - w \mid + \mid u - v \mid, \\ &(\mid v - w \mid) + \mid u - v \mid) \end{aligned}$$

$$\begin{aligned} &N((0, u, 0), (0, v, 0), (0, w, 0)) \\ &= ((\mid v - w \mid) + \mid u - v \mid, \\ &\frac{2}{3}(\mid v - w \mid) + \mid u - v \mid, \\ &(\mid v - w \mid + \mid u - v \mid)) \end{aligned}$$

$$\begin{aligned} &N((0, 0, u), (0, 0, y), (0, 0, w)) \\ &= ((\mid v - w \mid) + \mid u - v \mid, \\ &\mid v - w \mid + \mid u - v \mid, \\ &\frac{1}{3}(\mid v - w \mid + \mid u - v \mid)) \end{aligned}$$

and

$$\begin{aligned} &N((u, 0, 0), (0, v, 0), (0, 0, w)) \\ &= N((0, 0, w), (0, v, 0), (u, 0, 0), \\ &= \ldots \\ &= (u + v + \frac{1}{3}w, u + \frac{2}{3}v + w, \frac{4}{3}u + v + w) \end{aligned}$$

Then (X, N) *is a complete* N*-cone metric space. Let*

$$T : X \to X$$

given by

$$T(u, 0, 0) = T(0, u, 0) = (0, 0, \frac{1}{3}u)$$

and $T(0, 0, u) = (\frac{2}{3}u, 0, 0)$. *Because of this, the contractive condition in Theorem 24 is satisfied by* T, *with* $a = \frac{3}{4} \in [0, 1)$. *Thus* T *possesses a unique fixed point* $(0, 0, 0) \in X$.

Example 29. *Let* $E = \mathbb{R}, X = [0, +\infty)$ *and*

$$P = \{u \in R : u \geq 0\}$$

Define $N : X^3 \to E$ *by*

$$N(x, y, z) = \max\left\{ \mid v - w \mid, \mid u - w \mid, \mid u - v \mid \right\}$$

for all u, v, w *in* X.

It is obvious that the metric space (X, N) *is a complete* N*-cone space. Then*

$$N(u, 0, 0) = N(0, u, 0) = N(0, 0, u) = u$$

for all u *in* X.

We consider $T : X \to X$ *given by*

$$Tu = \frac{1}{6}u$$

for all $u \in X$. *Here,*

$$N(T\,u, Tv,\, T\,v) = N(\frac{1}{6}u, \frac{1}{6}\,v, \frac{1}{6}\,v) = \frac{1}{6} \mid u - v \mid .$$

and

$$\begin{aligned} &N(u, Tv, Tv) + N(v, Tu, Tu) \\ &= N(u, \frac{1}{6}v, \frac{1}{6}v) + N(v, \frac{1}{6}u, \frac{1}{6}u) \\ &= \mid u - \frac{1}{6}v \mid + \mid v - \frac{1}{6}u \mid . \end{aligned}$$

Take $a = \frac{2}{5} \in [0, \frac{1}{2}]$. *Then*

$$\frac{1}{6} \mid u - v \mid \leq a\left\{ \mid u - \frac{1}{6}v \mid + \mid v - \frac{1}{6}u \mid \right\}.$$

It follows that the contractive conditions of theorem 27 are satisfied by T. *As so,* T *has a unique fixed point* $0 \in X$.

V. Conclusions and Future Work

The primary objectives of this study are to establish a symmetric N-cone metric space and to determine the convergence conditions for sequences in this space. Additionally, we established many novel fixed point theorems in N-cone metric spaces in relation to these mappings. The results may be generalizable to other spaces with efficient constraints and should be useful in fixed point theory.

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