Partial Threshold Graphs

K ARATHI BHAT* and SHASHWATH S SHETTY

***Abstract*—Chain graphs and threshold graphs play a very important role in Spectral Graph Theory, since the maximizers for the largest eigenvalue of the adjacency matrix (for graphs of fixed order and size, either connected or disconnected) belong to these classes (threshold graphs in the general case, and chain graphs in the bipartite case). Nesting in the neighborhood of vertices in these graphs has gained the attention of various researchers. Motivated by this structure, we generalize and define a new class of graphs named it as 'partial threshold graphs' and study the properties. In this article, we give bounds and expressions for the Wiener index and Hyper-Wiener index of a partial threshold graph. We extend the study further and give a set of integers, except which every other integer is the Wiener index of some partial threshold graph. The highlight of the article is an algorithm for the inverse Wiener index problem of partial threshold graphs.**

***Index Terms*—Chain, bipartition, Wiener index, hyper-Wiener index, inverse Wiener index.**

I. INTRODUCTION

GRAPHS considered in this paper are simple, finite, undirected and connected with vertex set V(G) and edge set E(G). A collection $S = \{S_1, S_2, \cdots, S_n\}$ of sets is said to form a chain with respect to set inclusion, if for every $S_i, S_j \in S$ either $S_i \subseteq S_j$ or $S_j \subseteq S_i$. We write $u \sim v$ if the vertices u and v are adjacent in G, $u \nsim v$ if they are not. The neighborhood of the vertex $u \in V(G)$ is the set $N(u)$ consisting of all the vertices v such that $v \sim u$ in G. Readers are referred to [18] for all the elementary notations and definitions not described but used in this article.

Definition 1.1: A chain graph is a bipartite graph in which the neighborhoods of the vertices in each partite set form a chain with respect to set inclusion.

In other words, for every two vertices u and v in the same partite set and their neighborhoods $N(u)$ and $N(v)$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. We note that, every partite set in a chain graph has at least one dominating vertex, that is, a vertex adjacent to all the vertices of the other partite set. The color classes of a chain graph $G(V_1 \cup V_2, E)$ can be partitioned into h non-empty cells given by

$$V_1 = V_{11} \cup V_{12} \cup \ldots \cup V_{1h} \; and \; V_2 = V_{21} \cup V_{22} \cup \ldots \cup V_{2h}$$

such that $N(u) = V_{21} \cup V_{22} \cup \ldots \cup V_{2\ h-i+1}$, for any $u \in V_{1i},\ 1 \leq i \leq h$. If $m_i = |V_{1i}|$ and $n_i = |V_{2i}|$, then we write $G = DNG(m_1, \ldots, m_h; n_1, \ldots, n_h)$. Due to this nesting property, the chain graphs are also called double nested graphs (DNGs). If $m_i = n_i = 1$ for all $1 \leq i \leq h$, then the graph is called half graph.

A split graph is a graph which admits a partition of its vertex set into two parts W_1 and W_2 such that W_1 induces a complement of a clique (co-clique) and W_2 induces a clique. Every other edge, called a cross edge, joins a vertex of W_1 with a vertex of W_2. A threshold graph is a split graph in which the adjacencies defined by the cross edges satisfy the following nesting property. Both W_1 and W_2 can be partitioned into h cells, say,

$$W_1 = W_{11} \cup W_{12} \cup \ldots \cup W_{1h} \; and \; W_2 = W_{21} \cup W_{22} \cup \ldots \cup W_{2h}$$

such that $N(u) = W_{21} \cup W_{22} \ldots W_{2\ h-i+1}$, for any vertex $u \in W_{1i},\ 1 \leq i \leq h$. It is also called a nested split graph (NSG). If $m_i = |W_{1i}|$ and $n_i = |W_{2i}|$, then we write $G = NSG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$.

The readers are referred to [1], [2], [3] and [8] for more results on chain and threshold graphs.

Recently in 2022, motivated by the nesting property, the authors of the article [12] defined a partial chain graph (PCG) and studied its properties. Also, by extending the concept of nesting from a bipartite graph to a k-partite graph, a k-nested graph is defined in [16].

Motivated by the nesting property of these extremal graphs (chain and threshold graphs), we define a new class of graphs, whose vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that $\langle V_1 \rangle \cong \langle \overline{V_2} \rangle$ and has the nesting property. Formally, we define the same as follows.

Definition 1.2: A graph G on n vertices is said to be a partial threshold graph if its vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that the following conditions are satisfied.

1. $\langle V_1 \rangle \cong \langle \overline{V_2} \rangle$.
2. The set $\{V_i \cap N(v)\} \neq \phi$ form a chain with respect to set inclusion for every $v \in V_j, j \neq i, 1 \leq i, j \leq 2$.

We denote $N_1(u) = N(u) \cap V_1, u \in V_2$ and $N_2(v) = N(v) \cap V_2, v \in V_1$. The subsets V_1 and V_2 can be further partitioned into h cells $V_1 = V_{11} \cup \ldots \cup V_{1h}$ and $V_2 = V_{21} \cup \ldots \cup V_{2h}$ which satisfies the following

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nesting property:
For every vertex $u \in V_{1i}, 1 \leqslant i \leqslant h, N_2(u) = V_{21} \cup \ldots \cup V_{2\ h-i+1}$ and for $v \in V_{2j}, 1 \leqslant j \leqslant h$, $N_1(v) = V_{11} \cup \ldots \cup V_{1\ h-j+1}$. If $|V_{1i}| = m_i$ and $|V_{2i}| = n_i$, then we write $G = PTG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. From the condition 1 of Definition 1.2, it is clear that n must be even and $|V_1| = |V_2| = \frac{n}{2}$. When V_1 or V_2 is independent, we get a threshold graph.
Unlike the chain or threshold graphs, $G = PTG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ is not representing a single graph, instead it represents a graph family G_f with nesting property as explained earlier. It does not specify the structure of $\langle V_1 \rangle$ or $\langle V_2 \rangle$. Thus, we write $G_f = PTG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ (instead of just G). We use the notion $G \in G_f = PTG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ of graphs which have the bipartition $V(G) = V_1 \cup V_2$ such that $\langle V_1 \rangle \cong \langle \overline{V_2} \rangle$. Observe that the complete graph $K_2 = PTG(1;1) = NSG(1;1) = DNG(1;1)$ and $K_4 \setminus e = PTG(2;2) = NSG(2;2)$.

The graphs $G_1, G_2 \in G_f = PTG(1,2,1;1,1,2)$ as shown in Figure 1 are such that $\langle U_1 \rangle \ncong \langle V_1 \rangle$ and $\langle U_2 \rangle \ncong \langle V_2 \rangle$. Both the graphs G_1, G_2 have 8 vertices and 15 edges.

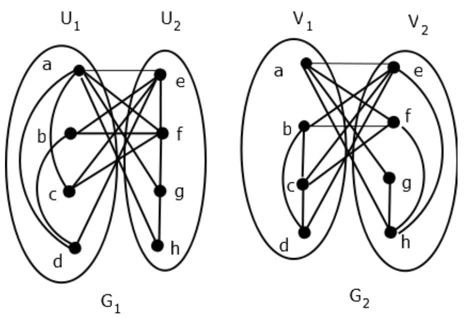

Fig. 1. $G_1, G_2 \in G_f = PTG(1,2,1;1,1,2)$

For the graph G_1, we observe that $\langle U_1 \rangle \cong \langle \overline{U_2} \rangle$ and for G_2, $\langle V_1 \rangle \cong \langle \overline{V_2} \rangle$.
The graphs $H_1, H_2 \in H_f = PTG(1,3;1,3)$ as shown in Figure 2 are such that $\langle U_1 \rangle \cong \langle V_1 \rangle$ and $\langle U_2 \rangle \cong \langle V_2 \rangle$, but they are not isomorphic to each other.

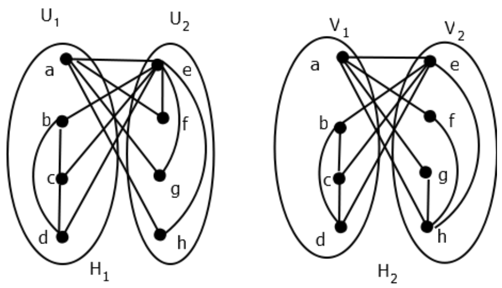

Fig. 2. $H_1, H_2 \in H_f = PTG(1,3;1,3)$

For all the 4 graphs G_1, G_2, H_1 and H_2, we observe that $N_2(d) \subseteq N_2(c) \subseteq N_2(b) \subseteq N_2(a)$ and $N_1(h) \subseteq N_1(g) \subseteq N_1(f) \subseteq N_1(e)$.

The rest of the paper is organized as follows; Section 2 deals with the properties of partial threshold graphs. A few bounds and expressions for the Wiener index of a partial threshold graph (strong partial threshold graph) are discussed in Section 3. About few results on Hyper-Wiener index is discussed in Section 4. The list of integers which would never be the Wiener indices of any strong partial threshold graph is obtained in Section 5. We conclude the article with the algorithm for the inverse Wiener index problem of strong partial threshold graphs.

II. Properties

In this section, we study few basic properties of a partial threshold graph.
By the definition of a partial threshold graph, it is clear that the partite set V_1 has at least one vertex, say u, such that $N_2(u) = V_2$, we call that vertex as dominating vertex in V_1 (dominating with respect to the other partite set). Similarly, V_2 also has at least one dominating vertex.
We note that when $n > 2$, none of the tree, complete graph, cycle graph is a partial threshold graph. All the graphs $G \in G_f = PTG(m_1, m_2, \ldots, m_h; n_2, n_2, \ldots, n_h)$ have the same number of edges and vertices. For further discussions we use $u_1, u_2, \ldots, u_{\frac{n}{2}}$ to denote the vertices of V_1 and $u'_1, u'_2, \ldots, u'_{\frac{n}{2}}$ to denote the vertices of V_2 in any partial threshold graph on n vertices.

Theorem 2.1: Let G be any partial threshold graph with the maximum degree $\Delta(G)$ and minimum degree $\delta(G)$. Then,

$$\frac{n}{2} + 1 \leq \Delta(G) \leq n - 1,$$
$$1 \leq \delta(G) \leq \frac{n}{2} + \left\lfloor \frac{n-2}{4} \right\rfloor .$$

Proof: The upper bound for the maximum degree, $\Delta(G)$ is attained for a dominating vertex u_i (or u'_i) of the set V_1 (or V_2) of a partial threshold graph when u_i (or u'_i) is adjacent to all other vertices of V_1 (or V_2). Any threshold graph with $|V_1| = |V_2| = \frac{n}{2}$ has $\Delta(G) = n-1$. The dominating vertex of any partial threshold graph is adjacent to $\frac{n}{2}$ vertices of other partite set. Suppose the dominating vertex say, $u_1 \in V_1$ is not adjacent to any vertex of V_1, then $u'_1 \in V_2$ is adjacent to u_1 of V_1, and to all the other vertices of V_2. In this case, the maximum degree is achieved by the vertex $u'_1 \in V_2$ and $deg(u'_1) = \frac{n}{2} + 1$.

It can be easily noted that a partial threshold graph can have pendant vertex. Hence, $\delta(G) = 1$. When $G \in G_f = PTG(\frac{n}{2}; \frac{n}{2})$ all the vertices are adjacent to $\frac{n}{2}$ vertices of other partite set. Apart from this any vertex v is adjacent to $\lfloor \frac{n-2}{4} \rfloor$ vertices in side the partite set where it belongs. Then $\delta(G) = \frac{n}{2} + \lfloor \frac{n-2}{4} \rfloor$. ■

Theorem 2.2: There exists a regular partial threshold graph on $n = 8k+2, k \geq 1$ vertices with regularity $\frac{n}{2} + \frac{n-2}{4}$.

Proof: Let $G \in G_f = PTG(\frac{n}{2}; \frac{n}{2})$. In order to get a regular partial threshold graph, degree of every vertex u_i in $\langle V_1 \rangle$ and u'_i in $\langle V_2 \rangle$ must be equal. This is possible only if both $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are $\left(\frac{n-2}{4}\right)$-regular, for which $\frac{n-2}{4}$ must be an integer (i.e., $n = 4k + 2, k \geq 1$ *or* $\frac{n}{2}$ should be odd). Since $\frac{n}{2}$ is odd and $\langle V_i \rangle, i = 1, 2$ is a regular graph of degree $\frac{n-2}{4}$ implies $\frac{n-2}{4}$ even. This implies $n = 8k + 2, k \geq 1$. Hence the proof. ■

Theorem 2.3: The diameter of any partial threshold graph G is 2 or 3.

Proof: Two vertices which are in same partite set are at distance at most 2, as either they are adjacent or commonly adjacent to a dominating vertex of another set. Consider any two non adjacent vertices $u_i \in V_1$ and $u'_j \in V_2$. If either u_i is adjacent to a dominating vertex of V_1, or u'_j is adjacent to a dominating vertex of V_2, then $d(u_i, u'_j) = 2$. If u_i is not adjacent to any dominating vertex of V_1 and u'_j is also not adjacent to any dominating vertex of V_2, then $d(u_i, u'_j) = 3$. Hence the diameter of G is 2 or 3. ■

In the next theorem, we characterize all partial threshold graphs with diameter equal to 2.

Theorem 2.4: A partial threshold graph G on n vertices is of diameter 2 if and only if any one of the following case is true.

1) $G \in G_f = PTG(\frac{n}{2}; \frac{n}{2})$.
2) G is a NSG with $|V_1| = |V_2| = \frac{n}{2}$.
3) For every pair of non adjacent vertices u_i, u'_j, either there should exists a vertex say, $u_k \in V_1$, such that $u_i \sim u_k$ and $u_k \sim u'_j$ or a vertex $u'_l \in V_2$ such that $u'_j \sim u'_l$ and $u'_l \sim u_i$.

Theorem 2.5: Let G be any partial threshold graph on n vertices and m edges. Then,

$$\binom{\frac{n}{2}}{2} + n - 1 \leq m \leq \binom{\frac{n}{2}}{2} + \frac{n^2}{4}.$$

Proof: We know that $|E(\langle V_1 \rangle)| + |E(\langle V_2 \rangle)| = \binom{\frac{n}{2}}{2}$. The minimum number of cross edges possible in a PTG is $n-1$ and hence the lower bound. Similarly, the maximum number of cross edges possible is $\frac{n^2}{4}$, and hence the upper bound. ■

The conditions for the addition of edges to a chain (threshold) graph G such that the resultant graph is also a chain (threshold) graph is given in [3] ([11]). The following theorem gives the conditions for the addition of the edges to a partial threshold graph G such that the resultant graph is also a partial threshold graph.

Theorem 2.6: Let $G \in G_f = PTG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a partial threshold graph on n vertices, where $m_j = |V_{1j}|$ *and* $n_j = |V_{2j}|$ for $1 \leq j \leq h$. The graph $G + e$ obtained by adding an edge $e = (u_k, u'_l)$ to G is a partial threshold graph if and only if $u_k \in V_{1i}$ and $u'_l \in V_{2\ h-i+2}$, for some $2 \leq i \leq h, 1 \leq k, l \leq \frac{n}{2}$.

III. Wiener Index

The Wiener index is one of the oldest and most studied topological indices, both from theoretical point of view and applications. Due to its strong connection to chemistry, where molecules have a tree-like structure, a lot of research was done on acyclic graphs (see [6] for survey). The Wiener index $W(G)$ of a graph G is the sum of all distances between all pairs of vertices in G.

$$W(G) = \sum_{\{u,v\} \in V(G)} d(u, v).$$

In this section, we give an expression as well as bounds for the Wiener index of partial threshold graphs.

Theorem 3.1: Let G be a partial threshold graph on $n = 2p$ vertices and size m. Then the Wiener index of G is given by

$$W(G) = 5\binom{p}{2} + 3p^2 - 2m - k, \qquad \text{(III.1)}$$

where k is number of pairs of vertices (a, b) with distance 2, such that $a \in V_1$ and $b \in V_2$.

Proof: The total number of pairs of vertices which are adjacent in either $\langle V_1 \rangle$ or in $\langle V_2 \rangle$ is equal to $\binom{p}{2}$. Hence, total number of pairs of vertices (a, b) which are at distance 2 such that either $a, b \in V_1$ or $a, b \in V_2$ is $\binom{p}{2}$. There are $p^2 - m + \binom{p}{2}$ pairs of non adjacent vertices such that the distance between them is either 2 or 3 with $a \in V_1$ and $b \in V_2$. Therefore, there are $p^2 - m + \binom{p}{2} - k$ pairs of vertices (a, b) which are at distance 3 with $a \in V_1$ and $b \in V_2$. Hence,

$$\begin{aligned} W(G) &= m + 2\left(\binom{p}{2} + k\right) + 3\left(p^2 - m + \binom{p}{2} - k\right) \\ &= 5\binom{p}{2} + 3p^2 - 2m - k. \end{aligned}$$

■

Corollary 3.2: Let G be a partial threshold graph on $n = 2p$ vertices and m edges with diameter 2. Then the Wiener index of G is given by

$$W(G) = 4\binom{p}{2} + 2p^2 - m. \qquad \text{(III.2)}$$

Proof: As G is of diameter 2, proof follows by substituting $k = p^2 - m + \binom{p}{2}$ in Equation III.1. ∎

The following corollary gives the bounds for the Wiener index of partial threshold graphs on $n = 2p$ vertices of diameter 2 in terms of p.

Corollary 3.3: Let G be a partial threshold graph on $n = 2p$ vertices and m edges with diameter 2. Then the Wiener index of G satisfies the relation

$$3\binom{p}{2} + p^2 \leq W(G) \leq 3\binom{p}{2} + 2p^2 - 2p + 1.$$

Proof: From Theorem 2.5, we have $2p-1+\binom{p}{2} \leq m \leq p^2 + \binom{p}{2}$. Proof follows by substituting these bounds of m in Equation (III.2). ∎

Next, we give the bounds for the Wiener index of any partial threshold graphs on $n = 2p$ vertices in terms of p.

Theorem 3.4: Let G be a partial threshold graph on $n = 2p$ vertices and size m. Then the Wiener index of G satisfies the relation

$$3\binom{p}{2} + p^2 \leq W(G) \leq \frac{9p^2 - 13p + 6}{2}.$$

Proof: The expression in Equation (III.1) is minimum when $2m+k$ is maximum i.e., when $m = p^2+\binom{p}{2}$ and $k = 0$. The lower bound is attained by substituting $m = p^2 + \binom{p}{2}$ and $k = 0$.
Similarly, the expression in Equation (III.1) is maximum when $2m+k$ is minimum. We know that the minimum value of m is $\binom{p}{2} + 2p - 1$. Also, $k \neq 0$, as among the $p^2 - 2p + 1$ pairs of vertices (a, b) which are not adjacent with $a \in V_1$ and $b \in V_2$, all pairs cannot be at distance 3.
Suppose the dominating vertex of V_1 say, u_1 is adjacent to at least one vertex $u_j \in V_1, 2 \leqslant j \leqslant p$, then $d(u_j, u'_k) = 2, 2 \leqslant k \leqslant p$. Hence the value of k is at least $p-1$. Suppose the dominating vertex of V_1 say, u_1 is not adjacent to any vertex $u_j \in V_1, 2 \leqslant j \leqslant p$, and u'_1 is not the dominating vertex of V_2. Then as u'_1 is adjacent to all the vertices of V_2, we have $d(u'_1, u_j) = 2, 2 \leqslant j \leqslant p$. Hence $k \geq p - 1$.

We show the existence of a partial threshold graph $H \in H_f = PTG(1, p-1; 1, p-1)$ with $k = p-1$. Without loss of generality assume that $u_1, \ldots, u_p$ and $u'_1, \ldots, u'_p$ are vertices of V_1 and V_2 respectively. Let u_1 and u'_p be the dominating vertex of V_1 and V_2 respectively, and $u'_1 \sim u'_j, 2 \leqslant j \leqslant p$ and $\langle u_2, \ldots, u_p \rangle \cong K_{p-1}$. The graph H attains the bound for k i.e., $k = p - 1$. Hence, substituting $k = p - 1$ and $m = \binom{p}{2} + 2p - 1$ in Equation (III.1), the upper bound of $W(G)$ is attained. ∎

Corollary 3.5: Let $G \in G_f = PTG(1, p - 1; 1, p - 1)$ be a partial threshold graph. Then, the Wiener index of G satisfies the relation

$$\frac{7p^2 - 7p + 2}{2} \leq W(G) \leq \frac{9p^2 - 13p + 6}{2}.$$

Proof: We know that the number of edges in G is $\binom{p}{2} + 2p-1$. The lower bound is attained when there is no pairs of vertices in G which are at distance 3. This can be attained when $G = NSG(1, p - 1; 1, p - 1)$. The upper bound is attained when $G = H \in H_f = PTG(1, p - 1; 1, p - 1)$ of Theorem 3.4. ∎

Corollary 3.6: Let $G \in G_f = PTG(p; p)$ be a partial threshold graph. Then,

$$W(G) = 3\binom{p}{2} + p^2.$$

Proof: Proof follows from the fact that G is of diameter 2 and substituting $m = \binom{p}{2} + p^2$ in Equation (III.2). ∎

For further results we impose some strong conditions on a partial threshold graph and name it as strong partial threshold graph. For a partial threshold graph $G(V_1 \cup V_2, E)$ with $V_1 = \{u_1, u_2, \ldots, u_p\}$ and $V_2 = \{u'_1, u'_2, \ldots, u'_p\}$, we have $N_2(u_i) = N(u_i) \cap V_2$ and $N_1(u'_i) = N(u'_i) \cap V_1, 1 \leq i \leqslant p$.

Definition 3.1: Consider a partial threshold graph $G(V_1 \cup V_2, E)$ with $V_1 = \{u_1, u_2, \ldots, u_p\}$ and $V_2 = \{u'_1, u'_2, \ldots, u'_p\}$ and $N_2(u_i) \subseteq N_2(u_{i-1}), 2 \leqslant i \leqslant p$. Then G is said to be a strong partial threshold graph if there exists a bijective mapping $\Phi : V_1 \to V_2$ satisfying the following conditions:
(i) $u_i \sim u_j$ implies $\Phi(u_i) \nsim \Phi(u_j)$, for all $1 \leq i \neq j \leq p$.
(ii) $N_1(\Phi(u_i)) \subseteq N_1(\Phi(u_{i-1})), 2 \leqslant i \leqslant p$.
Without loss of generality we denote $\Phi(u_i)$ by u'_i.
The nested split graphs with $|V_1| = |V_2|$ and $PTG(p; p)$ are also strong partial threshold graphs. But $PTG(1, p-1; 1, p-1)$ is strong PTG if and only if $\Phi(u) = v$, where $u \in V_1$ and $v \in V_2$ are the dominating vertices.
The graphs G_1 and G_2 of Figure 1 are strong partial threshold graphs with $\Phi(a) = e, \Phi(b) = f, \Phi(c) = g$ and $\Phi(d) = h$. Also the graph H_1 as shown in Figure 2 is strong PTG. But for H_2 of Figure 2, $\Phi(a) = h, \Phi(b) = g, \Phi(c) = f$ and $\Phi(d) = e$ and $N_2(d) \subseteq N_2(c) \subseteq N_2(b) \subseteq N_2(a)$. But as $N_1(e) \nsubseteq N_1(f) \nsubseteq N_1(g) \nsubseteq N_1(h)$, H_2 is not a strong PTG.

Let $u_1, u_2, \ldots, u_{\frac{n}{2}}$ be the vertices of V_1 and let $u'_1, u'_2, \ldots, u'_{\frac{n}{2}}$ be the vertices of V_2 in any strong partial threshold graph on n vertices. Then, $deg_{\langle V_1 \rangle}(u_i) + deg_{\langle V_2 \rangle}(u'_i) = \frac{n}{2} - 1$.

We obtain the bounds for the Wiener index of a strong partial threshold graphs in the following theorems.

Theorem 3.7: Let $G \in G_f = PTG(1, 1, \ldots, 1; 1, 1, \ldots, 1)$ be a strong partial threshold graph on $n = 2p$ vertices. Then, the Wiener index of G satisfies the relation

$$3p^2 - 2p \le W(G) \le \begin{cases} \frac{25p^2-14p-8}{8} & if \text{ p } even \\ \frac{25p^2-16p-1}{8} & if \text{ p } odd \end{cases}.$$

Proof: Lower bound is attained by a $NSG(1, 1, \ldots, 1; 1, 1, \ldots, 1)$.
To obtain upper bound we consider p even and odd cases separately. We obtain the minimum value of k of the Equation (III.1).

(i) p even : There are $p^2 - \dfrac{p(p+1)}{2}$ non adjacent pairs of vertices (a, b), such that $a \in V_1$ and $b \in V_2$. Let the vertices of G be labeled as in Definition 3.1. We know that either $u_1 \sim u_2$ or $u'_1 \sim u'_2$. Suppose $u_1 \sim u_2$ then $d(u_2, u'_p) = 2$, which implies $k \ge 1$. Similarly, if we assume $u_1 \sim u_j, 2 \leqslant j \leqslant \frac{p+2}{2}$, implies that $k \ge 1 + 2 + \ldots + \frac{p}{2}$. Also as $u'_1 \sim u'_j, \frac{p+4}{2} \leqslant j \leqslant p$, gives $d(u_r, u'_s) = 2, \frac{p+4}{2} \leqslant r, s' \leqslant p$. Hence $k \ge (1 + 2 + \cdots + \frac{p}{2}) + (\frac{p-2}{2})^2 = \frac{3p^2-6p+8}{8}$. Then, there are $p^2 - \frac{p(p+1)}{2} - \frac{3p^2-6p+8}{8}$ pairs of vertices (a, b) such that $d(a, b) = 3$, where $a \in V_1, b \in V_2$.
We show the existence of a strong PTG which satisfies $k = \frac{3p^2-6p+8}{8}$. Consider a strong partial threshold graph $H \in H_f = PTG(1, \ldots, 1; 1, \ldots, 1)$, with $u_1 \sim u_j, 2 \leqslant j \leqslant \frac{p+2}{2}$, and $u'_1 \sim u'_j, \frac{p+4}{2} \leqslant j \leqslant p$.
Suppose $\langle u_1, u_2, \ldots, u_{\frac{p+2}{2}} \rangle \cong K_{\frac{p+2}{2}}$ and $\langle u_{\frac{p+4}{2}}, \ldots, u_p \rangle \cong K_{\frac{p-2}{2}}$. Then, $d(u_i, u'_j) = 2, 2 \le i \le \frac{p+2}{2}, p - i + 2 \le j \le p$ and $d(u_r, u'_s) = 2, \frac{p+4}{2} \le r, s \le p$. Hence, $k = (1 + 2 + \cdots + \frac{p}{2}) + (\frac{p-2}{2})^2 = \frac{3p^2-6p+8}{8}$. Substituting the value of k in Equation (III.1), we get $W(G) = \frac{25p^2-14p-8}{8}$. Hence the upper bound.

(ii) p odd : Suppose $u_1 \sim u_j, 2 \leqslant j \leqslant \frac{p+1}{2}$, then $u'_1 \sim u'_j, \frac{p+3}{2} \leqslant j \leqslant p$. Then $k \ge 1 + 2 + \cdots + \frac{p-1}{2} + (\frac{p-1}{2})^2 = \frac{3p^2-4p+1}{8}$.
We show the existence of a strong partial threshold graph $H \in H_f = PTG(1, 1, \ldots, 1; 1, 1, \ldots, 1)$ satisfying $k = \frac{3p^2-4p+1}{8}$. Suppose in H, the graph induced by the vertices $\{u_1, u_2, \ldots, u_{\frac{p+1}{2}}\}$ is isomorphic to $K_{\frac{p+1}{2}}$ and $\langle u_{\frac{p+3}{2}}, \ldots, u_p \rangle \cong K_{\frac{p-1}{2}}$. Then, $d(u_i, u'_j) = 2, 2 \le i \le \frac{p+1}{2}, p - i + 2 \le j \le p$ and $d(u_r, u'_k) = 2, \frac{p+3}{2} \le r, k \le p$. Hence value of k is $1 + 2 + \cdots + \frac{p-1}{2} + (\frac{p-1}{2})^2 = \frac{3p^2-4p+1}{8}$.
Substituting the value of k in Equation (III.1), we get $W(G) = \frac{25p^2-16p-1}{8}$. Hence the upper bound. ■

Theorem 3.8: Let G be a strong partial threshold graph on $n = 2p$ vertices and size m. Then the Wiener index of G satisfies the relation

$$3\binom{p}{2} + p^2 \le W(G) \le \begin{cases} \frac{15p^2-16p+4}{4} & if \text{ p } even \\ \frac{15p^2-16p+5}{4} & if \text{ p } odd. \end{cases}$$

Proof: The expression in Equation (III.1) is minimum when $2m + k$ is maximum i.e., when $m = p^2 + \binom{p}{2}$ and $k = 0$. The lower bound is attained by a $NSG(p; p)$.

Similarly, Equation (III.1) attains maximum value, when $2m+k$ is minimum. We know that the minimum value of m is $\binom{p}{2} + 2p - 1$. We note that $k \neq 0$, as among the $p^2 - 2p + 1$ pairs of vertices (a, b) which are not adjacent with $a \in V_1$, $b \in V_2$, all pairs cannot be at distance 3. Let the vertices $u_i \in V_1$, and $u'_i \in V_2, 1 \le i \le p$ of G be labeled as in Definition 3.1. Consider $G \in G_f = PTG(1, p-1; 1, p-1)$. From condition (iii) of Theorem 2.4, if $u_i \sim u_1, 2 \leqslant i \leqslant p$ or $u'_i \sim u'_1, 2 \leqslant i \leqslant p$, then all the $p^2 - 2p + 1$ non adjacent pairs (a, b) of vertices with $a \in V_1, b \in V_2$ are at distance 2. We obtain the minimum value of k when p is even and odd separately.

(i) p even : Suppose $u_1 \sim u_i, 2 \leqslant i \leqslant \frac{p+2}{2}$ then $u'_1 \sim u'_j, \frac{p+4}{2} \leqslant j \leqslant p$. Then, $d(u_i, u'_j) = 2, 2 \le i \le \frac{p+2}{2}, 2 \le j \le p$ and $d(u_r, u'_s) = 2, \frac{p+4}{2} \le r, s \le p$. Hence value of k is $\frac{p}{2}(p-1) + (\frac{p-2}{2})^2 = \frac{3p^2-6p+4}{4}$.
We show the existence of a strong partial threshold graph $H \in H_f = PTG(1, p-1; 1, p-1)$ satisfying $k = \frac{3p^2-6p+4}{4}$. Let $u_1 \sim u_i, 2 \leqslant i \leqslant \frac{p+2}{2}$ and $u_{\frac{p+2}{2}+l} \sim u_{\frac{p+2}{2}+l+1}, 0 \le l \leqslant \frac{p}{2} - 2$. Then, H has $k = \frac{3p^2-6p+4}{4}$.
Substituting the value of k in Equation III.1, we get, $W(G) = \frac{15p^2-16p+4}{4}$. Hence, the upper bound.

(ii) p odd : Suppose $u_1 \sim u_j, 2 \leqslant j \leqslant \frac{p+1}{2}$ then $u'_1 \sim u'_j, \frac{p+3}{2} \leqslant j \leqslant p$. Then, $d(u_i, u'_j) = 2, 2 \le i \le \frac{p+1}{2}, 2 \le j \le p$ and $d(u_r, u'_s) = 2, \frac{p+3}{2} \le r, s \le p$. Hence, $k \ge \frac{p-1}{2}(p-1) + (\frac{p-1}{2})^2 = \frac{3p^2-6p+3}{4}$.
We show the existence of a strong partial threshold graph $H \in H_f = PTG(1, p-1; 1.p-1)$ satisfying $k = \frac{3p^2-6p+3}{4}$. Let $u_i \sim u_j, 2 \leqslant j \leqslant \frac{p+1}{2}$ and $u_{\frac{p+1}{2}+l} \sim u_{\frac{p+1}{2}+i+1}, 0 \leqslant l \leqslant \frac{p-3}{2}$. Then, H has $k = \frac{3p^2-6p+3}{4}$.
Substituting the value of k in Equation (III.1), we get, $W(G) = \frac{15p^2-16p+5}{4}$. Hence the upper bound. ■

IV. Hyper-Wiener index

The hyper-Wiener index of a graph G is defined [14] as

$$\begin{aligned} WW(G) &= \sum_{\{u,v\} \in V(G)} d(u, v) + \sum_{\{u,v\} \in V(G)} d(u, v)^2 \\ &= W(G) + \sum_{\{u,v\} \in V(G)} d(u, v)^2. \end{aligned}$$

The relation between Wiener index and hyper-Wiener index has been the subject of study in [9]. The different methods for calculating the hyper-Wiener index of molecular structures is discussed in [5], and also computed for the various operations of graphs in [13].

In this section, we give few results related to hyper-Wiener index of a partial threshold graph.

Theorem 4.1: Let G be a partial threshold graph on $n = 2p$ vertices and size m. Then the hyper-Wiener index of G is given by

$$WW(G) = 18\binom{p}{2} + 12p^2 - 10m - 6k, \qquad \text{(IV.3)}$$

where k is number of pairs of vertices (a, b) with distance 2, such that $a \in V_1$ and $b \in V_2$.

Proof: Proof follows from Theorem 3.1, and Equation III.1. ■

Corollary 4.2: Let G be a partial threshold graph on $n = 2p$ vertices and m edges with diameter 2. Then the hyper-Wiener index of G is given by

$$WW(G) = 12\binom{p}{2} + 6p^2 - 4m. \qquad \text{(IV.4)}$$

The following corollary gives the bounds for the hyper-Wiener index of partial threshold graphs on $n = 2p$ vertices of diameter 2 in terms of p.

Corollary 4.3: Let G be a partial threshold graph on $n = 2p$ vertices and m edges with diameter 2. Then the hyper-Wiener index of G satisfies the relation

$$8\binom{p}{2} + 2p^2 \leq WW(G) \leq 8\binom{p}{2} + 6p^2 - 8p + 4.$$

Theorem 4.4: Let G be a partial threshold graph on $n = 2p$ vertices and size m. Then the hyper-Wiener index of G satisfies the relation

$$6p^2 - 4p \leq WW(G) \leq 16p^2 - 30p + 16.$$

Proof: Lower bound is achieved by substituting $m = p^2 + \binom{p}{2}$ and $k = 0$ in Equation IV.3. Similarly, upper bound is achieved by substituting $m = \binom{p}{2} + 2p - 1$ and $k = p - 1$ in Equation IV.3. ■

Corollary 4.5: Let $G \in G_f = PTG(p; p)$ be a partial threshold graph. Then,

$$WW(G) = 6p^2 - 4p.$$

Corollary 4.6: Let $G \in G_f = PTG(1, p-1; 1, p-1)$ be a partial threshold graph. Then, the hyper-Wiener index of G satisfies the relation

$$10p^2 - 12p + 4 \leq WW(G) \leq 16p^2 - 30p + 16.$$

Proof: Lower bound follows by substituting $k = p^2 - 2p + 1$ and $m = \binom{p}{2} + 2p - 1$ in Equation IV.3 and the lower bound is achieved by $NSG(1, p-1; 1, p-1)$. ■

V. Inverse Wiener Index

The term inverse Wiener index problem refers to problem of constructing the graph of order n, given a Wiener index $W(G) = k$. It turned out that every positive integer, except for two and five, is the Wiener index of some connected graph. In 1995, Gutman and Ye [10] considered an inverse Wiener index problem as follows; For which integers n there exist trees with Wiener index n?

A list of integers which are forbidden values for the Wiener indices of connected bipartite graphs ([10]) and trees, unicyclic graphs ([17]) has appeared in the literature. Authors of the article [4] gave a list of integers which would never be the Wiener indices of any chain / threshold graphs. In this section, we list all the integers which are forbidden values for the Wiener indices of partial threshold graphs.

Theorem 5.1: Let $A = 3\binom{p}{2} + p^2$ and $B = \begin{cases} \frac{15p^2-16p+4}{4} & \text{if } p \text{ } is \text{ } even \\ \frac{15p^2-16p+5}{4} & if \text{ } p \text{ } is \text{ } odd \end{cases}$, where $p \geq 2$. Then for every integer $k \in [A, B]$, there exists at least one strong partial threshold graph on $n = 2p$ vertices.

Proof: By Corollary 3.3, if G is a partial threshold graph on $n = 2p$ vertices, size m and diameter 2, then the Wiener index of G satisfies the relation $3\binom{p}{2} + p^2 \leq W(G) \leq 3\binom{p}{2} + 2p^2 - 2p + 1$. The upper bound is also satisfied by $G = NSG(1, p-1; 1, p-1)$ (a strong threshold graph). Let the vertices $u_i \in V_1$, and $u'_i \in V_2, 1 \leq i \leq p$ of G be labeled as in Definition 3.1. Suppose if we add edges $u_i \sim u'_j, 2 \leq i, j \leq p$ one by one to $NSG(1, p-1; 1, p-1)$ satisfying the condition given in Theorem 2.6, we observe that, the Wiener index decreases by exactly one. Adding exactly $p^2 - 2p + 1$ edges one by one to $NSG(1, p-1; 1, p-1)$ we get a strong partial threshold graph which is a $NSG(p; p)$. The graph $NSG(p; p)$ has the least Wiener index given by $3\binom{p}{2} + p^2$. This implies every value in the interval $\left[3\binom{p}{2} + p^2, 3\binom{p}{2} + 2p^2 - 2p + 1\right]$ is a Wiener index of some strong partial threshold graph G. But the Wiener index of a strong partial threshold graph G satisfies the relation

$$3\binom{p}{2} + p^2 \leq W(G) \leq \begin{cases} \frac{15p^2-16p+4}{4} & \text{if } p \text{ } is \text{ } even \\ \frac{15p^2-16p+5}{4} & if \text{ } p \text{ } is \text{ } odd \end{cases}.$$

Next, we show the existence of a strong partial threshold graph whose Wiener index is every value in the interval $\left[3\binom{p}{2} + 2p^2 - 2p + 2, \frac{15p^2-16p+4}{4}\right]$ when p is even and $\left[3\binom{p}{2} + 2p^2 - 2p + 2, \frac{15p^2-16p+5}{4}\right]$ when p is odd.

(i) p even : We know that the upper bound is satisfied by a strong partial threshold graph $G = PTG(1, p-1; 1, p-1)$, in which $u_1 \sim u_j, 2 \leqslant j \leqslant \frac{p+2}{2}$ and $u_{\frac{p+2}{2}+i} \sim u_{\frac{p+2}{2}+i+1}, 0 \leq i \leq \frac{p}{2} - 2$.
Suppose if we add edges $u_i \sim u'_j, 2 \leq i \leq \frac{p}{2}, 2 \leq j \leq p$ one by one, we note that it satisfies the condition given in Theorem 2.6. Then by adding these $\left(\frac{p}{2} - 1\right)(p-1)$ edges one by one, the Wiener index decreases by exactly one. We

have $\frac{15p^2-16p+4}{4} - \left(3\binom{P}{2} + 2p^2 - 2p + 2\right) = \frac{p^2-2p-4}{4}$. As, $\frac{(p-1)(p-2)}{2} - \frac{p^2-2p-4}{4} = \frac{p^2-4p+8}{4} \geq 0$, it follows that every value in the interval $\left[3\binom{p}{2} + 2p^2 - 2p + 2, \frac{15p^2-16p+4}{4}\right]$ is a Wiener index of some strong partial threshold graph G.

(ii) p odd: We know that the upper bound is satisfied by a strong partial threshold graph $G = PTG(1, p-1; 1, p-1)$, in which $u_1 \sim u_j, 2 \leqslant j \leqslant \frac{p+1}{2}$ and $u_{\frac{p+1}{2}+l} \sim u_{\frac{p+1}{2}+i+1}, 0 \leq l \leq \frac{p-3}{2}$. Now suppose if we add edges continuously to G, such that $u_i \sim u'_j, 2 \leq i \leq \frac{p-1}{2}, 2 \leq j \leq p$ it satisfies the condition given in Theorem 2.6. We note that by adding these $\left(\frac{p-3}{2}\right)(p-1)$ edges one by one, the Wiener index decreases by exactly one. We have $\frac{15p^2-16p+5}{4} - \left(3\binom{P}{2} + 2p^2 - 2p + 2\right) = \frac{(p-3)(p+1)}{4}$. As, $\frac{(p-1)(p-3)}{2} - \frac{(p-3)(p+1)}{4} = \frac{(p-3)^2}{4} \geq 0$, it follows that every value in the interval $\left[3\binom{p}{2} + 2p^2 - 2p + 2, \frac{15p^2-16p+5}{4}\right]$ is a Wiener index of some partial threshold graph G. ■

For the sake of simplicity to address, we define realizability of a positive integer in the above said context. An integer $k \in Z^+$ is said to be realizable Wiener index for a partial threshold graph if there exists at least one partial threshold graph G with the Wiener index k. If not, we say k is forbidden. For every integer k within the respective bounds, the next theorem guarantees the existence of at least one partial threshold graph with the Wiener index k.

Theorem 5.2: An integer $k \in Z^+$ is realizable Wiener index for a strong partial threshold graph if and only if it satisfies the following condition; $k \in [3\binom{p}{2}+p^2, \frac{15p^2-16p+4}{4}]$, when p is even or $k \in [3\binom{p}{2} + p^2, \frac{15p^2-16p+5}{4}]$, when p is odd.

We characterize the integers which are forbidden to be the Wiener indices of any partial threshold graph. For all consecutive integers p and $p+1$ whenever $p \geq 6$, it is true that the upper bound of the Wiener index of a strong partial threshold graph for p is greater than or equal to the lower bound for $p+1$.
Thus, all the integers $k \geq 81$ are realizable Wiener indices. Further, for $p = 1, 2, 3, 4$ and 5 the bounds are $[1, 1]$, $[7, 8]$, $[18, 23]$, $[34, 45]$ and $[55, 75]$ respectively and all the integers in these intervals are realizable from Theorem 5.2. Thus, the integers $2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 46, 47, 48, 49, 50, 51, 52, 53, 54, 76, 77, 78, 79$ and 80 are forbidden to be the Wiener indices of strong partial threshold graph.

Theorem 5.3: Every integer except $2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 46, 47, 48, 49, 50, 51, 52, 53, 54, 76, 77, 78, 79$ and 80 is the Wiener index of some strong partial threshold graph G. As every strong partial threshold graph is also a partial threshold graph, from Theorem 5.2, all the integers $k \geqslant 81$ are realizable Wiener indices of a partial threshold graph also. For $p = 1, 2, 3, 4$ and 5 the bounds of Wiener indices of a partial threshold graphs are $[1, 1], [7, 8], [18, 24], [34, 49]$, and $[55, 83]$. The partial threshold graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$ and G_9 having the Wiener indices $24, 46, 47, 49, 76, 77, 78, 79$ and 80 respectively are shown in Figure 3.

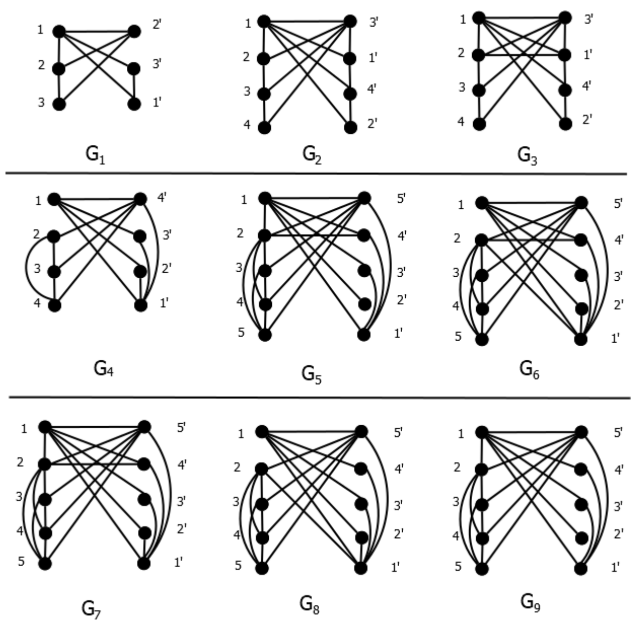

Fig. 3. Partial Threshold Graphs

With all this theory and conclusions, we now propose the main theorem of this article.

Theorem 5.4: Every integer except $2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 25, 26, 27, 28, 29, 30, 31, 32, 33, 48, 50, 51, 52, 53$ and 54 is the Wiener index of some partial threshold graph G.

VI. Algorithm

An algorithm which finds a tree with a given Wiener index can be found in [7]. Similarly, an algorithm that returns the chain graph (threshold graph) with the given Wiener index is discussed in [4], [15]. In this section we present an algorithm for the inverse Wiener index problem of partial threshold graphs. In the following we gave the bounds for number of vertices of a partial threshold graphs in terms of the Wiener index which we use in the algorithm to obtain possible p value.

Theorem 6.1: Let W be any given integer. There exist a strong partial threshold graph on $n = 2p$ vertices with the Wiener index W if p satisfies

$$\left\lceil \frac{8 + \sqrt{60W - 11}}{15} \right\rceil \leq p \leq \left\lfloor \frac{3 + \sqrt{40W + 9}}{10} \right\rfloor.$$

Proof: Proof follows from Theorem 3.8. ∎

Corollary 6.2: Let $W \notin \{24, 46, 47, 49, 76, 77, 78, 79, 80\}$ be any integer. Then there exist a partial threshold graph on $n = 2p$ vertices with the Wiener index W if p satisfies

$$\left\lceil \frac{8+\sqrt{60W-11}}{15} \right\rceil \leq p \leq \left\lfloor \frac{3+\sqrt{40W+9}}{10} \right\rfloor.$$

The graphs $G_{max} = NSG(1, p-1; 1, p-1)$ and $G_{min} = NSG(p; p)$ which we use in the algorithm has the split partition $V(G) = V_1 \cup V_2$ with $\langle V_1 \rangle$, $\langle V_2 \rangle$ as co-clique and clique respectively. The graph $G_1^*, G_2^* \in G_f = PTG(1, p-1; 1, p-1)$. The vertices of G, G_1^*, G_2^* are labeled as follows: $V_1 = \{1, 2, \ldots, p\}$ and $V_2 = \{1', 2', \ldots, p'\}$ with $\Phi(1) = 1'$. In G_1^*, $1 \sim j$, $2 \leqslant j \leqslant \frac{p+2}{2}$ and $k \sim k+1$, $\frac{p+2}{2} \leqslant k \leqslant p-1$ and in G_2^*, $1 \sim j$, $2 \leqslant j \leqslant \frac{p+1}{2}$ and $k \sim k+1$, $\frac{p+1}{2} \leqslant k \leqslant p-1$.

If the given input $k \in \{24, 46, 47, 49, 76, 77, 78, 19, 80\}$, then the algorithm returns the partial threshold graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9$ respectively, which are given in Figure 3. For a given Wiener index k if there does not exist any possible p value, then the algorithm prints 'There does not exist a PTG with the Wiener index k'. If $k \in [3\binom{p}{2} + p^2, 3\binom{p}{2} + 2p^2 - 2p + 1]$, then the algorithm adds the required number of edges to G_{max}, otherwise the algorithm add required number of edges to G_1^* or G_2^*, such that it results in a partial threshold graph with the given input as its Wiener index.

Algorithm 1 add_edges_NSG(G, p, C)

for $i = 2$ to p **do**
  for $j = 2'$ to p' **do**
    if $C \neq 0$ **then**
      $E(G) = E(G) \cup (i, j)$
      $C = C - 1$
    end if
  end for
end for
return G =0

Algorithm 2 add_edges_PTG(G, p, C)

for $i = 2$ *to* $\lfloor \frac{p}{2} \rfloor$ **do**
  for $j = 2'$ *to* p' **do**
    if $C \neq 0$ **then**
      $E(G) = E(G) \cup (i, j)$
      $C = C - 1$
    end if
  end for
end for
return G =0

Algorithm 3 PTGWiener(k)

0: Input k
0: Output A PTG with the given Wiener index if exists.
0: $INDICES = [24, 46, 47, 49, 76, 77, 78, 79, 80]$
1: **for** i in range 9 **do**
2:   **if** k == INDICES[i] **then**
3:     **return** G_i
4:   **end if**
5: **end for**
5: $p = \lceil \frac{8+\sqrt{60k-11}}{15} \rceil$
5: $q = \lfloor \frac{3+\sqrt{40k+9}}{10} \rfloor$
5: $M = 3\binom{p}{2} + 2p^2 - 2p + 1$
6: **if** $p > q$ **then**
7:   **return** "There does not exist a PTG with the Wiener index k"
8: **else**
9:   **if** $k == \frac{5p^2-3p}{2}$ **then**
10:     **return** G_{min}
11:   **else if** $k == M$ **then**
12:     **return** G_{max}
13:   **else if** $k \leq M$ **then**
13:     $C = M - k$
13:     $G = G_{max}$
14:     **return** $add\_edges\_NSG(G, p, C)$
15:   **else**
16:     $A = \frac{15p^2-16p+4}{4}$
17:     $B = \frac{15p^2-16p+5}{4}$
18:     **if** $k == A$ **then**
19:       **return** G_1^*
20:     **else if** $k == B$ **then**
21:       **return** G_2^*
22:     **else**
23:       **if** $p \ mod\ 2 == 0$ **then**
24:         $C = A - k$
25:         $G = G_1^*$
26:       **else**
27:         $C = B - k$
28:         $G = G_2^*$
29:       **end if**
30:       **return** add_edges_PTG(G, p, C)
31:     **end if**
32:   **end if**
33: **end if**=0

A. Illustrations

We illustrate the algorithm by taking particular values of k.

Example 6.1: Let $k = 8$.

1) It is clear that $8 \notin INDICES$. (lines 1-5 of Algorithm 3)
2) $p = q = 2$ and $M = 8$ (lines 5 of Algorithm 3);
3) $p \not> q$ and hence continue (lines 6-7 of Algorithm 3);
4) $\frac{5p^2-3p}{2} = 7 \neq k$ and as $k = M = 8$ and the algorithm returns G_{max} as output (line 8-12 of Algorithm 3).

Example 6.2: Let $k = 20$.

1) is clear that $20 \notin INDICES$ and hence continue (lines 1-5 of Algorithm 3);

2) $p = q = 3$ and $M = 22$ (line 5 of Algorithm 3)
3) $p \not> q$ and hence continue (lines 6-7 of Algorithm 3)
4) As $\frac{5p^2-3P}{2} = 18 \neq k$, $M = 22 \neq k$, and $k \leq 22$, $C = 22 - 20 = 2$ and set $G = G_{max}$ (minimal threshold graph on $2p$ vertices) hence return $add\_edges\_NSG(G_{max}, 3, 2)$ (lines 8-14 of Algorithm 3) and proceed to Algorithm 1.
5) Since $C = 2 \neq 0$ add the edge $(2, 2')$ to G and update the value of C with $C - 1 = 1$ (lines 1-5 of Algorithm 1).
6) Now, $C = 1 \neq 0$, add the edge $(2, 3')$ to G. By updating the value of C, we get $C = 0$ and hence it returns G (lines 1-9 of Algorithm 1) with $E(G) = \{E(G\_max) \cup (2, 2') \cup (2, 3')\}$.

B. Time Complexity

The maximum number of edges that can be added in Algorithm 1 is $(p-1)^2$ and in algorithm 2 it is $(\frac{p}{2}-1)(p-1)$. Hence, the time complexity of the Algorithm 1 and 2 is $\mathcal{O}(n^2)$. If the graph has been returned in the lines 1-5, the time required is constant, i.e., $\mathcal{O}(1)$. If there does not exist any partial threshold graph with the given integer as the Wiener index, Algorithm 3 stops in constant time. If the input value is equal to the Wiener index of G_{min} or G_{max}, then also the algorithm returns the graph in constant time. Suppose $k \leq M$, then the maximum value of C is equal to $(p-1)^2$ and hence the time required to return the graph is the constant times the time complexity of the function $add\_edges\_NSG(G, p, C)$ (Algorithm 1), that is equal to $\mathcal{O}(n^2)$. Suppose $k > M$ and either $k = A$ or $k = B$, the graph will be returned in constant time. Suppose $k > M$ and neither $k = A$ nor $k = B$, the time required to return the graph is constant times the time complexity of the function $add\_edges\_PTG(G, p, C)$ (Algorithm 2). Therefore, the time complexity of the algorithm is $\mathcal{O}(n^2)$.

VII. Conclusion

The highlight of the article is the list of integers which would never be the Wiener indices of any partial threshold graphs. Analogous to the algorithm for the inverse Wiener index problem for chain graphs and threshold graphs, we carry out a similar study and present an algorithm for partial threshold graphs. Further, other topological indices of a partial threshold graph can be studied.

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