Evaluating Mechanism and Axiomatic Processes under Sustainability and Multiple Criteria Fuzzy Behavior

Te-Chun Lee, Yen-Ching OuYang, Pang-Tung Liu, and Yu-Hsien Liao

***Abstract*—Due to the rapid global environmental changes, sustainability-related issues have garnered significant attention from various stakeholders. Addressing how to reduce and balance damages caused by various factors is an important focus of sustainability-related research. To this end, this paper first proposes several evaluating mechanisms for reducing and balancing damages under multiple criteria fuzzy behavior, and then adopts some axioms to analyze these mechanisms concurrently in terms of mathematical correctness and practical application by means of axiomatic processes. In order to modify the discrimination and relative effects caused by participants and its energy levels respectively, two weighted evaluating mechanisms and corresponding characterizations are also introduced. Besides, some more interpretations for these axioms and relative axiomatic processes are discussed throughout this paper.**

***Index Terms*—Sustainability, damage, evaluating mechanism, multiple criteria fuzzy behavior, axiomatic process.**

I. Introduction

Recently, sustainability-related issues have received heightened attention due to drastic climate change, depletion of available resources, and other environmental factors, leading to the emergence of related research on topics such as resource allocation, pollution mitigation, and warming suppression. The damage inflicted on the environment by civilization development has become an undeniable fact, with some damages even irreversible. Therefore, reducing damages caused by various factors has become a crucial aspect of sustainability-related research.

Reducing damages typically requires addressing multiple facets simultaneously, which may sometimes conflict. For instance, achieving highly efficient pollution reduction through certain measures or equipment without consuming excessive energy or resources, and without generating other types of pollution or waste, necessitates considering these multiple facets simultaneously in an optimal or balanced state. In the field of mathematics, multiple criteria optimization or equilibrium aims to achieve such benefits within any operational system.

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Under conventional transferable-utility (TU) conditions, each participant is typically categorized as either fully engaged or entirely uninvolved with certain other participants. However, in the majority of cases, the level of participant engagement remains ambiguous and challenging to determine. Under *fuzzy* TU *situations* (Aubin [1], [2]), participants are allowed to engage with infinite variations in energy levels. Various evaluating mechanisms for fuzzy TU games have been widely applicable across numerous domains, as explored in works by Branzei et al. [5], Nishizaki and Sakawa [28], Muto et al. [27], Hwang [12], Li and Zhang [19], Meng and Zhang [25], Khorram et al. [17], Borkotokey and Mesiar [4], Hwang and Liao [15], Masuya and Inuiguchi [24], among others.

Consistency is a fundamental property of evaluating mechanisms within axiomatic techniques for traditional situations. It ensures that a value remains independent when certain participants are fixed with their assigned payoffs. This property asserts that recommendations made for any problem should align with those made in subproblems where the payoffs of specific participants are determined. Consistency has been defined in various ways depending on how the payoffs of participants who ”exit the bargaining” are defined. This property has been extensively examined across diverse topics through the application of *reduced situations*, including bargaining and cost allocation issues. Utilizing marginal contributions, the equal allocation of non-separable costs (EANSC, Ransmeier [30]) and the normalized index are suggested for traditional TU situations, as per the respective proposals. Moulin [26] demonstrated the concept of complement-reduction to illustrate that the EANSC could offer an equitable method for distributing utilities.

The findings presented above prompt a key inquiry:

- whether the marginal index and its associated outcomes could be expanded to address sustainability under multiple criteria fuzzy behavior.

In this context, we aim to establish the necessary mathematical foundations for evaluating multiple criteria optimally to analyze sustainability-related problems with multiple objectives under fuzzy behavior simultaneously. Departing from the frameworks of traditional and fuzzy TU situations, we consider the framework of *multiple criteria fuzzy TU situations*. Two new evaluations are firstly presented in Section 2: the *infinitesimal equal evaluation of non-separable damages* (IEENSD) and the *normalized marginal-index*. The evaluating concept IEENSD involves participants receiving infinitesimal marginal damages from operational coalitions and then evaluating the remaining damages equally. Conversely, the normalized marginal-index evaluates damage proportionally by applying the infinitesimal marginal damages of all participants to operational coalitions. These evaluations generalize the concept of marginal damages to account for fuzzy behavior and multiple criteria situations.

To justify these evaluations, we introduce an extended reduction and related properties of consistency, considered in Sections 3 and 4:

- The IEENSD is the only evaluation satisfying the properties of *multiple criteria standardness for situations* and *multiple criteria bilateral consistency*.
- The IEENSD is the only evaluation satisfying the properties of *multiple criteria efficiency*, *multiple criteria zero-independence*, *multiple criteria symmetry* and *multiple criteria bilateral consistency*.
- While the normalized marginal-index violates multiple criteria bilateral consistency, it adheres to the properties of *normalized-standardness of situations* and *specific bilateral consistency*.

Building upon the IEENSD concept, each participant initially receives infinitesimal marginal damages from operational coalitions and subsequently evaluates the remaining damages equally. This entails equal evaluation of any additional fixed damage (e.g., the cost of a common facility) among the relevant participants. However, the participants and its energy levels may vary across different scenarios. Under practical applications, the IEENSD may appear unrealistic due to varying participant sizes or bargaining abilities. Additionally, asymmetry may emerge when modeling different bargaining abilities among participants and energy levels. To address these issues, we propose different evaluations whereby any additional fixed damage is evaluated among participants and its energy levels proportionally to its weights. To mitigate discrimination and relative effects caused by participants and its energy levels, assigning weights to both the "participants" and its "energy levels" is a reasonable approach. In Section 5, we introduce the *weight function for participants* and the *weight function for levels*, facilitating two weighted extensions of the IEENSD and associated axiomatic processes. Throughout the study, additional interpretations and discussions regarding these axioms and axiomatic processes are provided to further elucidate its implications.

II. PRELIMINARIES

Let $\mathbb{P}$ denote the set of participants. For each participant $i \in \mathbb{P}$ and $e_i \in (0,1]$, we define $E_i = [0, e_i]$ as the energy level space of participant i, with $E_i^+ = (0, e_i]$ indicating active participation, and 0 indicating non-participation. Let $P \subseteq \mathbb{P}$ and $E^P = \prod_{i\in P} E_i$ denote the Cartesian product set of energy level spaces for participants in P. For any $K \subseteq P$, a participant coalition $K \subseteq P$ corresponds canonically to the fuzzy coalition $e^K \in E^P$, where $e_i^K = 1$ if $i \in K$ and $e_i^K = 0$ if $i \in P \setminus K$. Let 0_P represent the zero vector in $\mathbb{R}^P$. For $m \in \mathbb{N}$, 0_m denotes the zero vector in $\mathbb{R}^m$, and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **fuzzy transferable-utility (TU) situation**[1] is characterized as a triple (P, e, d), where P denotes a non-empty and finite set of participants, $e = (e_i)_{i\in P} \in (0,1]^P$ represents the vector indicating the highest levels of energy for each participant, and $d : E^P \to \mathbb{R}$ is a function satisfying $d(0_P) = 0$, assigning the worth that participants can obtain when operating at corresponding energy levels $\eta = (\eta_i)_{i\in P} \in E^P$. A **multiple criteria fuzzy TU situation** is defined as a triple (P, e, D^m), where $m \in \mathbb{N}$, $D^m = (d^t)_{t\in\mathbb{N}_m}$, and (P, e, d^t) represents a fuzzy TU situation for all $t \in \mathbb{N}_m$. The class encompassing all multiple criteria fuzzy TU situations is denoted as $\mathbb{MF}$.

An **evaluation** is defined as a mapping τ that assigns to each $(P, e, D^m) \in \mathbb{MF}$ an element

$$\tau\big(P, e, D^m\big) = \big(\tau^t\big(P, e, D^m\big)\big)_{t\in\mathbb{N}_m},$$

where $\tau^t\big(P, e, D^m\big) = \big(\tau_i^t\big(P, e, D^m\big)\big)_{i\in P} \in \mathbb{R}^P$ and $\tau_i^t\big(P, e, D^m\big)$ represents the payoff of participant i when i engages in $\big(P, e, d^t\big)$. For $(P, e, D^m) \in \mathbb{MF}$, $H \subseteq P$, and $\eta \in \mathbb{R}^P$, $NE(\eta) = \{i \in P | \eta_i \neq 0\}$ is defined to denote the set of participants with non-zero energy levels, and $\eta_H \in \mathbb{R}^H$ represents the restriction of η to H. For a given $i \in P$, the notation η_{-i} is introduced to denote $\eta_{P\setminus\{i\}}$, and $\mu = (\eta_{-i}, t) \in \mathbb{R}^P$ is defined by $\mu_{-i} = \eta_{-i}$ and $\mu_i = t$.

Next, we provide two generalized indexes under multiple criteria fuzzy behavior.

Definition 1:

1) The **infinitesimal equal evaluation of non-separable damages (IEENSD)**, $\overline{\Delta}$, is defined by

$$\begin{aligned} &\overline{\Delta_i^t}(P, e, D^m) \\ = \;& \Delta_i^t(P, e, D^m) + \tfrac{1}{|P|} \cdot \Big[d^t(e) - \sum_{k\in P} \Delta_k^t(P, e, D^m)\Big] \end{aligned}$$

for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$. The quantity $\Delta_i^t(P, e, D^m) = \inf_{j\in E_i^+}\{d^t(e_{-i}, j) - d^t(e_{-i}, 0)\}$ represents the **infinitesimal marginal damage** incurred by participant i in (P, e, d^t). From this point onward, our focus will be limited to bounded fuzzy TU situations, defined as those situations (P, e, d^t) for which there exists $M_v \in \mathbb{R}$ such that $d^t(\eta) \leq M_v$ for all $\eta \in E^P$. This restriction ensures that $\Delta_i(P, e, d^t)$ is well-defined. Within the framework of $\overline{\Delta}$, each participant initially receives their infinitesimal marginal damages, following which the remaining damages are allocated equally among them.

2) The **normalized marginal-index**, $\overline{\Gamma}$, is defined by

$$\overline{\Gamma_i^t}(P, e, D^m) = \frac{d^t(e)}{\sum\limits_{k\in P} \Delta_k^t(P, e, D^m)} \cdot \Delta_i^t(P, e, D^m)$$

for all $(P, e, D^m) \in \mathbb{MF}^*$, for all $t \in \mathbb{N}_m$ and for all $i \in P$, where $\mathbb{MF}^* = \{(P, e, D^m) \in \mathbb{MF} \mid \sum_{i\in P} \Delta_i^t(P, e, D^m) \neq 0 \text{ for all } t \in \mathbb{N}_m\}$. Under the concept of $\overline{\Gamma}$, all participants distribute the damage of the overall fuzzy coalition proportionally by utilizing the infinitesimal marginal damages of all participants.

Here we present a concise application of multiple criteria fuzzy TU situations in the context of "management". Such problems can be formulated as follows. Consider a

[1]A fuzzy TU situation, defined by Aubin [1], [2], is represented as a pair (P, d^a), where P is a coalition and v^a is a mapping such that $d^a : [0,1]^P \longrightarrow \mathbb{R}$ and $d^a(0_P) = 0$.

set $P = \{1, 2, \cdots, n\}$ representing all participants in a comprehensive management system (P, e, D^m). The function d^t serves as a damage function, assigning a value to each level vector $\eta = (\eta_i)_{i\in P} \in E^P$, indicating the benefits participants can achieve when each participant i engages in an operational strategy $\eta_i \in E_i$ within the sub-management system (P, e, d^t). Conceptualized in this manner, the grand management system (P, e, D^m) can be viewed as a multiple criteria fuzzy TU situation, where d^t represents each characteristic function and E_i denotes the set of all operational strategies available to participant i. In subsequent sections, we aim to demonstrate that the IEENSD and the normalized fuzzy marginal evaluation can offer "optimal evaluation mechanisms" for all participants, ensuring that the organization can derive benefits from each combination of operational strategies across multiple criteria fuzzy behaviors.

III. Axiomatic results for the IEENSD

In order to to analyze the rationality for the IEENSD, an extended reduction and some axioms are applied to present some axiomatic processes. An evaluation τ satisfies **multiple criteria efficiency (MCEFF)** if for all $(P, e, D^m) \in \mathbb{MF}$ and for all $t \in \mathbb{N}_m$, $\sum_{i\in P} \tau_i^t(P, e, D^m) = d^t(e)$. An evaluation τ satisfies **multiple criteria standardness for situations (MCSS)** if $\tau(P, e, D^m) = \overline{\Delta}(P, e, D^m)$ for all $(P, e, D^m) \in \mathbb{MF}$ with $|P| \leq 2$. An evaluation τ satisfies **multiple criteria symmetry (MCSYM)** if $\tau_i(P, e, D^m) = \tau_k(P, e, D^m)$ for all $(P, e, D^m) \in \mathbb{MF}$ with $\Delta_i^t(P, e, D^m) = \Delta_k^t(P, e, D^m)$ for some $i, k \in P$ and for all $t \in \mathbb{N}_m$. An evaluation τ satisfies **multiple criteria zero-independence (MCZI)** if $\tau(P, e, D^m) = \tau(P, e, Q^m) + (h^t)_{t\in\mathbb{N}_m}$ for all $(P, e, D^m), (P, e, Q^m) \in \mathbb{MF}$ with $d^t(\eta) = q^t(\eta) + \sum_{i\in NE(\eta)} h_i^t$ for some $h^t \in \mathbb{R}^P$, for all $t \in \mathbb{N}_m$ and for all $\eta \in E^P$.

Property MCEFF stipulates that all participants allocate the entire damage comprehensively. Property MCSS extends the two-person standardness axiom introduced by Hart and Mas-Colell [11]. Property MCSYM states that if the infinitesimal marginal damages are equal, then the payoffs should also be equal. Property MCZI can be viewed as an extremely weak form of *additivity*. As per Definition 1, it is evident that the IEENSD satisfies MCEFF, MCSS, MCSYM, and MCZI.

Moulin [26] introduced the concept of reduced situations, where each coalition in the subgroup can achieve payoffs for its members only if they align with the initial payoffs of "all" members outside the subgroup. A natural extension of Moulin's reduction to multiple criteria fuzzy TU situations can be defined as follows.

Let $(P, e, D^m) \in \mathbb{MF}$, $H \subseteq P$ and τ be an evaluation. The **reduced situation** $(S, e_H, D_{H,\tau}^m)$ is defined by $D_{H,\tau}^m = (d_{H,\tau}^t)_{t\in\mathbb{N}_m}$ and for all $\eta \in E^H$,

$$
\begin{aligned}
&d_{H,\tau}^t(\eta)\\
=&\begin{cases} 0 & \text{if } \eta = 0_H, \\ d^t\big(\eta, e_{P\setminus H}\big) - \sum\limits_{i\in P\setminus H} \tau_i^t(P, e, D^m) & \text{otherwise.} \end{cases}
\end{aligned}
$$

An evaluation τ adheres to the principle of multiple criteria bilateral consistency (MCBCIY) if $\tau_i^t(H, e_H, D_{H,\tau}^m) = \tau_i^t(P, e, D^m)$ for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, for all $H \subseteq P$ with $|H| = 2$, and for all $i \in H$.

Lemma 1: The IEENSD $\overline{\Delta}$ satisfies MCBCIY.

Proof: Let $(P, e, D^m) \in \mathbb{MF}$, $H \subseteq P$ and $t \in \mathbb{N}_m$. Assume that $|P| \geq 2$ and $|H| = 2$. Therefore,

$$
\begin{aligned}
&\overline{\Delta_i^t}(H, e_H, D_{H,\overline{\Delta}}^m)\\
=\;& \Delta_i^t(H, e_H, D_{H,\overline{\Delta}}^m)\\
&+\tfrac{1}{|H|}\cdot\Big[d_{H,\overline{\Delta}}^t(e_H) - \sum_{k\in H}\Delta_k^t(H, e_H, D_{H,\overline{\Delta}}^m)\Big]
\end{aligned}
\tag{1}
$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. Furthermore,

$$
\begin{aligned}
&\Delta_i^t(H, e_H, D_{H,\overline{\Delta}}^m)\\
=\;& \inf_{j\in E_i^+}\{d_{H,\overline{\Delta}}^t(e_{H\setminus\{i\}}, j) - d_{H,\overline{\Delta}}^t(e_{H\setminus\{i\}}, 0)\}\\
=\;& \inf_{j\in E_i^+}\{d^t(e_{-i}, j) - d^t(e_{-i}, 0)\}\\
=\;& \Delta_i^t(P, e, D^m).
\end{aligned}
\tag{2}
$$

By equations (1), (2) and definitions of $d_{H,\overline{\Delta}}^t$ and $\overline{\Delta}$,

$$
\begin{aligned}
&\overline{\Delta_i^t}(H, e_H, D_{H,\overline{\Delta}}^m)\\
=\;& \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|}\cdot\Big[d_{H,\overline{\Delta}}^t(e_H) - \sum_{k\in H}\Delta_k^t(P, e, D^m)\Big]\\
=\;& \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|}\cdot\Big[d^t(e) - \sum_{k\in P\setminus H}\overline{\Delta_k^t}(P, e, D^m)\\
&\qquad - \sum_{k\in H}\Delta_k^t(P, e, D^m)\Big]\\
=\;& \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|}\cdot\Big[\sum_{k\in H}\overline{\Delta_k^t}(P, e, D^m)\\
&\qquad - \sum_{k\in H}\Delta_k^t(P, e, D^m)\Big]\\
&\big(\textbf{by MCEFF of } \overline{\Delta}\big)\\
=\;& \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|}\cdot\Big[\tfrac{|H|}{|P|}\cdot\Big[d^t(e)\\
&\qquad - \sum_{k\in P}\Delta_k^t(P, e, D^m)\Big]\Big]\\
=\;& \Delta_i^t(P, e, D^m) + \tfrac{1}{|P|}\cdot\Big[d^t(e) - \sum_{k\in P}\Delta_k^t(P, e, D^m)\Big]\\
=\;& \overline{\Delta_i^t}(P, e, D^m)
\end{aligned}
$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. So, the IEENSD satisfies MCBCIY. ■

Next, we characterize the IEENSD by means of multiple criteria bilateral consistency.

Theorem 1: The IEENSD is the only evaluation satisfying MCSS and MCBCIY.

Proof: By Lemma 1, $\overline{\Delta}$ satisfies MCBCIY. Clearly, $\overline{\Delta}$ satisfies MCSS.

To prove uniqueness, suppose τ satisfies MCSS and MCBCIY. By MCSS and MCBCIY of τ, it is easy to derive that τ also satisfies MCEFF, hence we omit it. Let $(P, e, D^m) \in \mathbb{MF}$. By MCSS of τ, $\tau(P, e, D^m) = \overline{\Delta}(P, e, D^m)$ if $|P| \leq 2$. The case $|P| > 2$: Let $i \in P$, $t \in \mathbb{N}_m$ and $H = \{i, k\}$ for some $k \in P \setminus \{i\}$.

$$\begin{aligned} & \tau_i^t(P,e,D^m) - \tau_k^t(P,e,D^m) \\ = & \tau_i^t(H,e_H,D^m_{H,\tau}) - \tau_k^t(H,e_H,D^m_{H,\tau}) \\ & \big(\textbf{by MCBCIY of } \tau\big) \\ = & \overline{\Delta_i^t}(H,e_H,D^m_{H,\tau}) - \overline{\Delta_k^t}(H,e_H,D^m_{H,\tau}) \\ & \big(\textbf{by MCSS of } \tau\big) \\ = & \Delta_i^t(H,e_H,D^m_{H,\tau}) - \Delta_k^t(H,e_H,D^m_{H,\tau}) \\ = & \inf_{j\in E_i^+}\{d^t_{H,\tau}(e_{H\setminus\{i\}},j) - d^t_{H,\tau}(e_{H\setminus\{i\}},0)\} \\ & \quad - \inf_{j\in E_k^+}\{d^t_{H,\tau}(e_{H\setminus\{k\}},j) - d^t_{H,\tau}(e_{H\setminus\{k\}},0)\} \\ = & \inf_{j\in E_i^+}\{d^t(e_{-i},j) - d^t(e_{-i},0)\} \\ & \quad - \inf_{j\in E_k^+}\{d^t(e_{-k},j) - d^t(e_{-k},0)\} \\ = & \Delta_i^t(P,e,D^m) - \Delta_k^t(P,e,D^m) \\ = & \overline{\Delta_i^t}(P,e,D^m) - \overline{\Delta_k^t}(P,e,D^m). \end{aligned}$$

Thus,

$$\begin{aligned} & \tau_i^t(P,e,D^m) - \tau_k^t(P,e,D^m) \\ = & \overline{\Delta_i^t}(P,e,D^m) - \overline{\Delta_k^t}(P,e,D^m). \end{aligned}$$

By MCEFF of τ and $\overline{\Delta}$,

$$\begin{aligned} & |P|\cdot\tau_i^t(P,e,D^m) - d^t(e) \\ = & \sum_{k\in P}[\tau_i^t(P,e,D^m) - \tau_k^t(P,e,D^m)] \\ = & \sum_{k\in P}[\overline{\Delta_i^t}(P,e,D^m) - \overline{\Delta_k^t}(P,e,D^m)] \\ = & |P|\cdot\overline{\Delta_i^t}(P,e,D^m) - d^t(e). \end{aligned}$$

Hence, $\tau_i^t(P,e,D^m) = \overline{\Delta_i^t}(P,e,D^m)$ for all $i \in P$ and for all $t \in \mathbb{N}_m$. ■

Next, we characterize the IEENSD by means of related properties of MCEFF, MCSYM, MCZI and MCBCIY.

Lemma 2: If an evaluation τ satisfies MCEFF, MCSYM and MCZI, then τ satisfies MCSS.

Proof: Assume that an evaluation τ satisfies MCEFF, MCSYM and MCZI. Let $(P,e,D^m) \in \mathbb{MF}$. The proof is completed by MCEFF of τ if $|P| = 1$. Let $(P,e,D^m) \in \mathbb{MF}$ with $P = \{i,k\}$ for some $i \neq k$. We define a situation (P,e,Q^m) to be that $q^t(\eta) = d^t(\eta) - \sum_{i\in NE(\eta)} \Delta_i^t(P,e,D^m)$ for all $\eta \in E^P$ and for all $t \in \mathbb{N}_m$. By definition of Q^m,

$$\begin{aligned} & \Delta_i^t(P,e,Q^m) \\ = & \inf_{j\in E_i^+}\{q^t(j,e_k) - q^t(0,e_k)\} \\ = & \inf_{j\in E_i^+}\{d^t(j,e_k) - d^t(0,e_k) - \Delta_i^t(P,e,D^m)\} \\ = & \inf_{j\in E_i^+}\{d^t(j,e_k) - d^t(0,e_k)\} - \Delta_i^t(P,e,D^m) \\ = & \Delta_i^t(P,e,D^m) - \Delta_i^t(P,e,D^m) \\ = & 0. \end{aligned}$$

Similarly, $\Delta_k^t(P,e,Q^m) = 0$. Therefore, $\Delta_i^t(P,e,Q^m) = \Delta_k^t(P,e,Q^m)$. By MCSYM of τ, $\tau_i^t(P,e,Q^m) = \tau_k^t(P,e,Q^m)$. By MCEFF of τ,

$$q^t(e) = \tau_i^t(P,e,Q^m) + \tau_k^t(P,e,Q^m) = 2\cdot\tau_i^t(P,e,Q^m).$$

Therefore,

$$\begin{aligned} & \tau_i^t(P,e,Q^m) \\ = & \tfrac{q^t(e)}{2} \\ = & \tfrac{1}{2}\cdot\big[d^t(e) - \Delta_i(P,e,D^m) - \Delta_k(P,e,D^m)\big]. \end{aligned}$$

By MCZI of τ,

$$\begin{aligned} & \tau_i^t(P,e,D^m) \\ = & \Delta_i^t(P,e,D^m) + \tfrac{1}{2}\cdot\big[d^t(e) - \Delta_i^t(P,e,D^m) \\ & \qquad - \Delta_k^t(P,e,D^m)\big] \\ = & \overline{\Delta_i^t}(P,e,D^m). \end{aligned}$$

Similarly, $\tau_k^t(P,e,D^m) = \overline{\Delta_k^t}(P,e,D^m)$. Hence, τ satisfies MCSS. ■

Theorem 2: On $\mathbb{MF}$, the IEENSD is the only evaluation satisfying MCEFF, MCSYM, MCZI and MCBCIY.

Proof: By Definition 1, $\overline{\Delta}$ satisfies MCEFF, MCSYM and MCZI. The remaining proofs follow from Theorem 1 and Lemmas 1, 2. ■

The subsequent examples illustrate the logical independence of each axiom utilized in Theorems 1 and 2 from the remaining axioms.

Example 1: Define an evaluation τ by for all $(P,e,D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$,

$$\tau_i^t(P,e,D^m) = \begin{cases} \overline{\Delta_i^t}(P,e,D^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ satisfies MCSS, but it violates MCBCIY.

Example 2: Define an evaluation τ to be that

$$\tau_i^t(P,e,D^m) = \Delta_i^t(P,e,D^m)$$

for all $(P,e,D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$. Clearly, τ satisfies MCSYM, MCZI and MCBCIY, but it violates MCEFF and MCSS.

Example 3: Define an evaluation τ to be that

$$\tau_i^t(P,e,D^m) = \frac{d^t(e)}{|P|}$$

for all $(P,e,D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$. Clearly, τ satisfies MCEFF, MCSYM and MCBCIY, but it violates MCZI.

Example 4: Define an evaluation τ by for all $(P,e,D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$,

$$\begin{aligned} \tau_i^t(P,e,D^m) = & \big[d^t(e) - d^t(e_{-i},0)\big] + \tfrac{1}{|P|}\cdot\Big[d^t(e) \\ & - \sum_{k\in P}\big[d^t(e) - d^t(e_{-k},0)\big]\Big]. \end{aligned}$$

Clearly, τ satisfies MCEFF, MCZI and MCBCIY, but it violates MCSYM.

Example 5: Define an evaluation τ by for all $(P,e,D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$,

$$\begin{aligned} & \tau_i^t(P,e,D^m) \\ = & \Delta_i^t(P,e,D^m) + \tfrac{f^t(i)}{\sum_{k\in P} f^t(k)}\cdot\big[d^t(e) - \sum_{k\in P}\Delta_k^t(P,e,D^m)\big], \end{aligned}$$

where for all $(P,e,D^m) \in \mathbb{MF}$, $f^t : P \to \mathbb{R}^+$ is defined by $f^t(i) = f^t(k)$ if $\Delta_i^t(P,e,D^m) = \Delta_k^t(P,e,D^m)$. Define an evaluation ψ by for all $(P,e,D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$,

$$\psi_i^t(P,e,D^m) = \begin{cases} \overline{\Delta_i^t}(P,e,D^m) & \text{if } |P| \leq 2, \\ \tau_i^t(P,e,D^m) & \text{otherwise.} \end{cases}$$

Clearly, ψ satisfies MCEFF, MCSYM and MCZI, but it violates MCBCIY.

IV. The Axiomatic Results for the Normalized Marginal-Index

Similar to Theorem 1, we aim to characterize the normalized marginal-index through the lens of multiple criteria bilateral consistency. However, it becomes apparent that $(H, e_H, D^m_{H,\tau})$ does not exist when $\sum_{i\in H}\Delta^t_i(P,e,D^m) = 0$. Consequently, we introduce the concept of specific bilateral consistency (SBCIY) as follows. An evaluation τ adheres to specific bilateral consistency (SBCIY) if $(H, e_H, D^m_{H,\tau}) \in \mathbb{MF}^*$ for some $(P,e,D^m) \in \mathbb{MF}$ and for some $H \subseteq P$ with $|H| = 2$, such that $\tau^t_i(H, e_H, D^m_{H,\tau}) = \tau^t_i(P,e,D^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in H$.

Lemma 3: The normalized marginal-index satisfies SBCIY on $\mathbb{MF}^*$.

Proof: Let $(P,e,D^m) \in \mathbb{MF}^*$. If $|P| \le 2$, then the proof is completed. Assume that $|P| \ge 3$ and $H \subseteq P$ with $|H| = 2$. Similar to equation (2),

$$\Delta^t_i(H, e_H, D^m_{H,\overline{\Gamma}}) = \Delta^t_i(P,e,D^m). \tag{3}$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. Define that $\sigma^t = \frac{d^t(e)}{\sum_{p\in P}\Delta^t_p(P,e,D^m)}$. For all $i \in H$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned}
&\overline{\Gamma}^t_i(H, e_H, D^m_{H,\overline{\Gamma}}) \\
=\;& \frac{d^t_{H,\overline{\Gamma}}(e_H)}{\sum_{k\in H}\Delta^t_k(H,e_H,D^m_{H,\overline{\Gamma}})}\cdot\Delta^t_i(H,e_H,D^m_{H,\overline{\Gamma}}) \\
=\;& \frac{d^t(e) - \sum_{h\in P\setminus H}\overline{\Gamma}^t_h(P,e,D^m)}{\sum_{k\in H}\Delta^t_k(P,e,D^m)}\cdot\Delta^t_i(P,e,D^m) \\
& \big(\textbf{by equation (3) and definition of } D^m_{H,\overline{\Gamma}}\big) \\
=\;& \frac{\sum_{h\in H}\overline{\Gamma}^t_h(P,e,D^m)}{\sum_{k\in H}\Delta^t_k(P,e,D^m)}\cdot\Delta^t_i(P,e,D^m) \\
& \big(\textbf{by MCEFF of } \overline{\Gamma}\big) \\
=\;& \sigma^t\cdot\Delta^t_i(P,e,D^m) \\
& \big(\textbf{by Definition 1}\big) \\
=\;& \overline{\Gamma}^t_i(P,e,D^m). \\
& \big(\textbf{by Definition 1}\big)
\end{aligned} \tag{4}$$

By equations (3), (4), the evaluation $\overline{\Gamma}$ satisfies SBCIY. ■

An evaluation τ satisfies **normalized-standardness for situations (NSS)** if $\tau(P,e,d) = \overline{\Gamma}(P,e,d)$ for all $(P,e,d) \in \mathbb{MF}$, $|P| \le 2$.

Theorem 3: On $\mathbb{MF}^*$, the evaluation $\overline{\Gamma}$ is the only evaluation satisfying NSS and SBCIY.

Proof: By Lemma 3, $\overline{\Gamma}$ satisfies SBCIY. Clearly, $\overline{\Gamma}$ satisfies NSS.

To prove uniqueness, suppose τ satisfies SBCIY and NSS on $\mathbb{MF}^*$. By NSS and SBCIY of τ, it is easy to derive that τ also satisfies MCEFF, hence we omit it. Let $(P,e,D^m) \in \mathbb{MF}^*$. We will complete the proof by induction on $|P|$. If $|P| \le 2$, it is trivial that $\tau(P,e,D^m) = \overline{\Delta}(P,e,D^m)$ by NSS. Assume that it holds if $|P| \le p-1$, $p \le 3$. The case $|P| = p$: Let $i,j \in P$ with $i \ne j$ and $t \in \mathbb{N}_m$. By Definition 1, $\overline{\Delta^t_k}(P,e,D^m) = \frac{d^t(e)}{\sum_{h\in P}\Delta^t_h(P,e,D^m)}\cdot\Delta^t_k(P,e,D^m)$ for all $k \in P$. Assume that $\eta^t_k = \frac{\Delta^t_k(P,e,D^m)}{\sum_{h\in P}\Delta^t_h(P,e,D^m)}$ for all $k \in P$. Therefore,

$$\begin{aligned}
&\tau^t_i(P,e,D^m) \\
=\;& \tau^t_i\big(P\setminus\{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big) \\
& \big(\textbf{by SBCIY of } \tau\big) \\
=\;& \overline{\Delta}^t_i\big(P\setminus\{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big) \\
& \big(\textbf{by NSS of } \tau\big) \\
=\;& \frac{v^t_{P\setminus\{j\},\tau}(e_{P\setminus\{j\}})}{\sum_{k\in P\setminus\{j\}}\Delta^t_k\big(P\setminus\{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big)} \\
& \cdot\Delta^t_i\big(P\setminus\{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big) \\
=\;& \frac{d^t(e) - \tau^t_j(P,e,D^m)}{\sum_{k\in P\setminus\{j\}}\Delta^t_k(P,e,D^m)}\cdot\Delta^t_i(P,e,D^m) \\
& \big(\textbf{by equation (2)}\big) \\
=\;& \frac{d^t(e) - \tau^t_j(P,e,D^m)}{-\Delta^t_j(P,e,D^m) + \sum_{k\in P}\Delta^t_k(P,e,D^m)}\cdot\Delta^t_i(P,e,D^m).
\end{aligned} \tag{5}$$

By equation (5),

$$\tau^t_i(P,e,D^m)\cdot[1-\eta^t_j] = [d^t(e) - \tau^t_j(P,e,D^m)]\cdot\eta^t_j$$

$$\begin{aligned}
\Longrightarrow\;& \sum_{i\in P}\tau^t_i(P,e,D^m)\cdot[1-\eta^t_j] \\
& = [d^t(e) - \tau^t_j(P,e,D^m)]\cdot\sum_{i\in P}\eta^t_j \\
\Longrightarrow\;& d^t(e)\cdot[1-\eta^t_j] = [d^t(e) - \tau^t_j(P,e,D^m)]\cdot 1 \\
& \big(\textbf{by MCEFF of } \tau\big) \\
\Longrightarrow\;& d^t(e) - d^t(e)\cdot\eta^t_j = d^t(e) - \tau^t_j(P,e,D^m) \\
\Longrightarrow\;& \overline{\Delta^t_j}(P,e,D^m) = \tau^t_j(P,e,D^m).
\end{aligned}$$

The proof is completed. ■

The subsequent examples illustrate the logical independence of each axiom utilized in Theorem 3 from the remaining axioms.

Example 6: Define an evaluation τ to be that for all $(P,e,D^m) \in \mathbb{MF}^*$, for all $t \in \mathbb{N}_m$ and for all $i \in P$,

$$\tau^t_i(P,e,D^m) = 0.$$

Clearly, τ satisfies SBCIY, but it violates NSS.

Example 7: Define an evaluation τ to be that for all $(P,e,D^m) \in \mathbb{MF}^*$, for all $t \in \mathbb{N}_m$ and for all $i \in P$,

$$\tau^t_i(P,e,D^m) = \begin{cases} \overline{\Gamma}^t_i(P,e,D^m) & \text{, if } |P| \le 2, \\ 0 & \text{, otherwise.} \end{cases}$$

Clearly, τ satisfies NSS, but it violates SBCIY.

Remark 1: It is easy to show that the normalized marginal-index satisfies MCEFF, MCSYM and NSS, but it violates MCZI.

V. Two Weighted Extensions

In various scenarios, participants and their energy levels may be assigned distinct weights. These weights serve as *a-priori measures of importance*, capturing considerations beyond those represented by the characteristic function. For instance, in evaluating costs among investment projects, weights might correspond to the profitability of each project. Similarly, when distributing travel costs among visited institutions, as discussed by Shapley [32], weights could represent the duration of stay at each institution.

If $f : U \to \mathbb{R}^+$ be a positive function, then f is called a **weight function for participants**. If $w : E^U \to \mathbb{R}^+$ be a positive function, then w is called a **weight function for levels**. By these two types of the weight function, two weighted revisions of the IEENSD is defined as follows.

Definition 2:

- The **1-infinitesimal weighted evaluation of non-separable damages (1-IWENSD)**, Γ^f, is defined by for all $(P,e,D^m) \in \mathbb{MF}$, for all weight function for participants f, for all $t \in \mathbb{N}_m$ and for all participant $i \in P$,

$$\begin{aligned}\Gamma_i^{f,t}(P,e,D^m) &= \Delta_i^t(P,e,D^m) + \frac{f(i)}{\sum_{k\in P} f(k)} \cdot \big[d^t(e) \\ &\quad - \sum_{k\in P} \Delta_k^t(P,e,D^m)\big]. \end{aligned} \tag{6}$$

- The **2-infinitesimal weighted evaluation of non-separable damages (2-IWENSD)**, Γ^w, is defined by for all $(P,e,D^m) \in \mathbb{MF}$, for all weight function for participants w, for all $t \in \mathbb{N}_m$ and for all participant $i \in P$,

$$\begin{aligned}\Gamma_i^{w,t}(P,e,D^m) &= \Delta_i^{w,t}(P,e,D^m) + \frac{1}{|P|} \cdot \big[d^t(e) \\ &\quad - \sum_{k\in P} \Delta_k^{w,t}(P,e,D^m)\big], \end{aligned} \tag{7}$$

where $\Delta_i^{w,t}(P,e,D^m) = \inf_{j\in E_i^+} \{w(j) \cdot [d^t(e_{-i},j) - d^t(e_{-i},0)]\}$.

An evaluation τ satisfies **1-weighted standardness for situations (1WSS)** if $\tau(P,e,D^m) = \Gamma^f(P,e,D^m)$ for all $(P,e,D^m) \in \mathbb{MF}$ with $|P| \le 2$ and for all weight function for participants f. An evaluation τ satisfies **2-weighted standardness for situations (2WSS)** if $\tau(P,e,D^m) = \Gamma^w(P,e,D^m)$ for all $(P,e,D^m) \in \mathbb{MF}$ with $|P| \le 2$ and for all weight function for levels w. Similar to the proofs of Lemma 1 and Theorem 1, we propose the analogies of Lemma 1 and Theorem 1.

Lemma 4: The 1-IWENSD Γ^f and the 2-IWENSD Γ^w satisfy MCEFF simultaneously.

Proof: Let $(P,e,D^m) \in \mathbb{MF}$, f be a weight function for participants, w be a weight function for levels and $t \in \mathbb{N}_m$.

$$\begin{aligned} &\sum_{i\in P} \Gamma_i^{f,t}(P,e,D^m) \\ =\;& \sum_{i\in P} \Big[\Delta_i^t(P,e,D^m) \\ &\quad + \frac{f(i)}{\sum_{k\in P} f(k)} \cdot \big[d^t(e) - \sum_{k\in P} \Delta_k^t(P,e,D^m)\big]\Big] \\ =\;& \sum_{i\in P} \Delta_i^t(P,e,D^m) \\ &\quad + \frac{\sum_{i\in P} f(i)}{\sum_{k\in P} f(k)} \cdot \big[d^t(e) - \sum_{k\in P} \Delta_k^t(P,e,D^m)\big] \\ =\;& \sum_{i\in P} \Delta_i^t(P,e,D^m) + d^t(e) - \sum_{k\in P} \Delta_k^t(P,e,D^m) \\ =\;& d^t(e). \end{aligned}$$

So, the 1-IWENSD satisfies MCEFF. Further,

$$\begin{aligned} &\sum_{i\in P} \Gamma_i^{w,t}(P,e,D^m) \\ =\;& \sum_{i\in P} \Big[\Delta_i^{w,t}(P,e,D^m) \\ &\quad + \frac{1}{|P|} \cdot \big[d^t(e) - \sum_{k\in P} \Delta_k^{w,t}(P,e,D^m)\big]\Big] \\ =\;& \sum_{i\in P} \Delta_i^{w,t}(P,e,D^m) \\ &\quad + \frac{|P|}{|P|} \cdot \big[d^t(e) - \sum_{k\in P} \Delta_k^{w,t}(P,e,D^m)\big] \\ =\;& \sum_{i\in P} \Delta_i^{w,t}(P,e,D^m) + d^t(e) - \sum_{k\in P} \Delta_k^{w,t}(P,e,D^m) \\ =\;& d^t(e). \end{aligned}$$

So, the 2-IWENSD satisfies MCEFF. ■

Lemma 5: The 1-IWENSD Γ^f and the 2-IWENSD Γ^w satisfy MCBCIY simultaneously.

Proof: Let $(P,e,D^m) \in \mathbb{MF}$, $H \subseteq P$, f be a weight function for participants, w be a weight function for levels and $t \in \mathbb{N}_m$. Assume that $|P| \ge 2$ and $|H| = 2$. Therefore,

$$\begin{aligned} &\Gamma_i^{f,t}(H,e_H,D^m_{H,\Gamma^f}) \\ =\;& \Delta_i^t(H,e_H,D^m_{H,\Gamma^f}) \\ &\quad + \frac{f(i)}{\sum_{k\in H} f(k)} \big[d^t_{H,\Gamma^f}(e_H) - \sum_{k\in H} \Delta_k^t(H,e_H,D^m_{H,\Gamma^f})\big] \end{aligned} \tag{8}$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. Furthermore,

$$\begin{aligned} &\Delta_i^t(H,e_H,D^m_{H,\Gamma^f}) \\ =\;& \inf_{j\in E_i^+} \{d^t_{H,\Gamma^f}(e_{H\setminus\{i\}},j) - d^t_{H,\Gamma^f}(e_{H\setminus\{i\}},0)\} \\ =\;& \inf_{j\in E_i^+} \{d^t(e_{-i},j) - d^t(e_{-i},0)\} \\ =\;& \Delta_i^t(P,e,D^m). \end{aligned} \tag{9}$$

By equations (8), (9) and definitions of d^t_{H,Γ^f} and Γ^f,

$$\begin{aligned} &\Gamma_i^{f,t}(H,e_H,D^m_{H,\Gamma^f}) \\ =\;& \Delta_i^t(P,e,D^m) \\ &\quad + \frac{f(i)}{\sum_{k\in H} f(k)} \big[d^t_{H,\Gamma^f}(e_H) - \sum_{k\in H} \Delta_k^t(P,e,D^m)\big] \\ =\;& \Delta_i^t(P,e,D^m) \\ &\quad + \frac{f(i)}{\sum_{k\in H} f(k)} \big[d^t(e) - \sum_{k\in P\setminus H} \Gamma_k^{f,t}(P,e,D^m) \\ &\quad - \sum_{k\in H} \Delta_k^t(P,e,D^m)\big] \\ =\;& \Delta_i^t(P,e,D^m) + \frac{f(i)}{\sum_{k\in H} f(k)} \big[\sum_{k\in H} \Gamma_k^{f,t}(P,e,D^m) \\ &\quad - \sum_{k\in H} \Delta_k^t(P,e,D^m)\big] \\ &\big(\textbf{by MCEFF of } \overline{\Delta}\big) \\ =\;& \Delta_i^t(P,e,D^m) + \frac{f(i)}{\sum_{k\in H} f(k)} \Big[\frac{\sum_{k\in H} f(k)}{\sum_{b\in P} f(b)} \big[d^t(e) \\ &\quad - \sum_{b\in P} \Delta_b^t(P,e,D^m)\big]\Big] \\ =\;& \Delta_i^t(P,e,D^m) + \frac{f(i)}{\sum_{b\in P} f(b)} \big[d^t(e) - \sum_{b\in P} \Delta_b^t(P,e,D^m)\big] \\ =\;& \Gamma_i^{f,t}(P,e,D^m) \end{aligned}$$

for all $i \in H$, for all weight function for participants f and for all $t \in \mathbb{N}_m$. So, the 1-IWENSD satisfies MCBCIY. Further, assume that $|P| \ge 2$ and $|H| = 2$. Therefore,

$$\begin{aligned} &\Gamma_i^{w,t}(H,e_H,D^m_{H,\Gamma^w}) \\ =\;& \Delta_i^{w,t}(H,e_H,D^m_{H,\Gamma^w}) \\ &\quad + \frac{1}{|H|} \big[d^t_{H,\Gamma^w}(e_H) - \sum_{k\in H} \Delta_k^{w,t}(H,e_H,D^m_{H,\Gamma^w})\big] \end{aligned} \tag{10}$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. Furthermore,

$$\begin{aligned} &\Delta_i^{w,t}(H,e_H,D^m_{H,\Gamma^w}) \\ =\;& \inf_{j\in E_i^+} \{w(j)[d^t_{H,\Gamma^w}(e_{H\setminus\{i\}},j) - d^t_{H,\Gamma^w}(e_{H\setminus\{i\}},0)]\} \\ =\;& \inf_{j\in E_i^+} \{w(j)[d^t(e_{-i},j) - d^t(e_{-i},0)]\} \\ =\;& \Delta_i^{w,t}(P,e,D^m). \end{aligned} \tag{11}$$

By equations (10), (11) and definitions of d^t_{H,Γ^w} and Γ^w,

$$\begin{aligned}
& \Gamma^{w,t}_i(H, e_H, D^m_{H,\Gamma^w}) \\
= \ & \Delta^{w,t}_i(P, e, D^m) \\
& + \tfrac{1}{|H|}\big[d^t_{H,\Gamma^w}(e_H) - \sum_{k\in H} \Delta^{w,t}_k(P, e, D^m)\big] \\
= \ & \Delta^{w,t}_i(P, e, D^m) \\
& + \tfrac{1}{|H|}\big[d^t(e) - \sum_{k\in P\setminus H} \Gamma^{w,t}_k(P, e, D^m) \\
& \qquad - \sum_{k\in H} \Delta^{w,t}_k(P, e, D^m)\big] \\
= \ & \Delta^{w,t}_i(P, e, D^m) + \tfrac{1}{|H|}\big[\sum_{k\in H} \Gamma^{w,t}_k(P, e, D^m) \\
& \qquad - \sum_{k\in H} \Delta^{w,t}_k(P, e, D^m)\big] \\
& \big(\textbf{by MCEFF of } \overline{\Delta}\big) \\
= \ & \Delta^{w,t}_i(P, e, D^m) + \tfrac{1}{|H|}\Big[\tfrac{|H|}{|P|}\big[d^t(e) \\
& \qquad - \sum_{b\in P} \Delta^{w,t}_b(P, e, D^m)\big]\Big] \\
= \ & \Delta^{w,t}_i(P, e, D^m) + \tfrac{1}{|P|}\big[d^t(e) - \sum_{b\in P} \Delta^{w,t}_b(P, e, D^m)\big] \\
= \ & \Gamma^{w,t}_i(P, e, D^m)
\end{aligned}$$

for all $i \in H$, for all weight function for levels w and for all $t \in \mathbb{N}_m$. So, the 2-IWENSD satisfies MCBCIY. ■

Remark 2: By Definition 2, it is easy to show that the 1-IWENSD violates MCSYM. Similarly, the 2-IWENSD violates MCSYM and MCZI.

Theorem 4:

- On $\mathbb{MF}$, the 1-IWENSD Γ^f is the only evaluation satisfying 1WSS and MCBCIY.
- On $\mathbb{MF}$, the 2-IWENSD Γ^w is the only evaluation satisfying 2WSS and MCBCIY.

Proof: By Lemma 5, Γ^f and Γ^w satisfy MCBCIY simultaneously. Clearly, Γ^f and Γ^w satisfy 1WSS and 2WSS respectively.

To prove the uniqueness of result 1, suppose τ satisfies 1WSS and MCBCIY. By 1WSS and MCBCIY of τ, it is easy to derive that τ also satisfies MCEFF, hence we omit it. Let $(P, e, D^m) \in \mathbb{MF}$ and f be a weight function for participants. By 1WSS of τ, $\tau(P, e, D^m) = \Gamma^f(P, e, D^m)$ if $|P| \le 2$. The case $|P| > 2$: Let $i \in P$, $t \in \mathbb{N}_m$ and $H = \{i, k\}$ for some $k \in P \setminus \{i\}$.

$$\begin{aligned}
& \tau^t_i(P, e, D^m) - \Gamma^{f,t}_i(P, e, D^m) \\
= \ & \tau^t_i(H, e_H, D^m_{H,\tau}) - \Gamma^{f,t}_i(H, e_H, D^m_{H,\Gamma^f}) \\
& \big(\textbf{by MCBCIY of } \tau \textbf{ and } \Gamma^f\big) \\
= \ & \Gamma^{f,t}_i(H, e_H, D^m_{H,\tau}) - \Gamma^{f,t}_i(H, e_H, D^m_{H,\Gamma^f}) \\
& \big(\textbf{by 1WSS of } \tau\big) \\
= \ & \frac{f(i)}{\sum_{b\in H} f(b)}\big[d^t_{H,\tau}(e_H) - d^t_{H,\Gamma^f}(e_H)\big] \\
& \big(\textbf{similar to equation (9)}\big) \\
= \ & \frac{f(i)}{\sum_{b\in H} f(b)}\big[\tau^t_i(P, e, D^m) + \tau^t_k(P, e, D^m) \\
& \qquad -\Gamma^{f,t}_i(P, e, D^m) - \Gamma^{f,t}_k(P, e, D^m)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
& f(k)\big[\tau^t_i(P, e, D^m) - \Gamma^{f,t}_i(P, e, D^m)\big] \\
= \ & f(i)\big[\tau^t_k(P, e, D^m) - \Gamma^{f,t}_k(P, e, D^m)\big].
\end{aligned}$$

By MCEFF of τ and Γ^f,

$$\begin{aligned}
& \frac{\sum_{k\in P} f(k)}{f(i)} \cdot \big[\tau^t_i(P, e, D^m) - \Gamma^{f,t}_i(P, e, D^m)\big] \\
= \ & \sum_{k\in P} \big[\tau^t_k(P, e, D^m) - \Gamma^{f,t}_k(P, e, D^m)\big] \\
= \ & d^t(e) - d^t(e) \\
= \ & 0.
\end{aligned}$$

Hence, $\tau^t_i(P, e, D^m) = \Gamma^{f,t}_i(P, e, D^m)$ for all $i \in P$, for all weight function for participants f and for all $t \in \mathbb{N}_m$. To prove the uniqueness of result 2, suppose τ satisfies 2WSS and MCBCIY. By 2WSS and MCBCIY of τ, it is easy to derive that τ also satisfies MCEFF, hence we omit it. Let $(P, e, D^m) \in \mathbb{MF}$ and w be a weight function for levels. By 2WSS of τ, $\tau(P, e, D^m) = \Gamma^w(P, e, D^m)$ if $|P| \le 2$. The case $|P| > 2$: Let $i \in P$, $t \in \mathbb{N}_m$ and $H = \{i, k\}$ for some $k \in P \setminus \{i\}$.

$$\begin{aligned}
& \tau^t_i(P, e, D^m) - \Gamma^{w,t}_i(P, e, D^m) \\
= \ & \tau^t_i(H, e_H, D^m_{H,\tau}) - \Gamma^{w,t}_i(H, e_H, D^m_{H,\Gamma^w}) \\
& \big(\textbf{by MCBCIY of } \tau \textbf{ and } \Gamma^w\big) \\
= \ & \Gamma^{w,t}_i(H, e_H, D^m_{H,\tau}) - \Gamma^{w,t}_i(H, e_H, D^m_{H,\Gamma^w}) \\
& \big(\textbf{by 2WSS of } \tau\big) \\
= \ & \tfrac{1}{|H|}\big[d^t_{H,\tau}(e_H) - d^t_{H,\Gamma^w}(e_H)\big] \\
& \big(\textbf{similar to equation (11)}\big) \\
= \ & \tfrac{1}{2}\big[\tau^t_i(P, e, D^m) + \tau^t_k(P, e, D^m) \\
& \qquad -\Gamma^{w,t}_i(P, e, D^m) - \Gamma^{w,t}_k(P, e, D^m)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
& \big[\tau^t_i(P, e, D^m) - \Gamma^{w,t}_i(P, e, D^m)\big] \\
= \ & \big[\tau^t_k(P, e, D^m) - \Gamma^{w,t}_k(P, e, D^m)\big].
\end{aligned}$$

By MCEFF of τ and Γ^w,

$$\begin{aligned}
& |P| \cdot \big[\tau^t_i(P, e, D^m) - \Gamma^{w,t}_i(P, e, D^m)\big] \\
= \ & \sum_{k\in P} \big[\tau^t_k(P, e, D^m) - \Gamma^{w,t}_k(P, e, D^m)\big] \\
= \ & d^t(e) - d^t(e) \\
= \ & 0.
\end{aligned}$$

Hence, $\tau^t_i(P, e, D^m) = \Gamma^{w,t}_i(P, e, D^m)$ for all $i \in P$, for all weight function for levels w and for all $t \in \mathbb{N}_m$. ■

The following examples are to show that each of the axioms used in Theorem 4 is logically independent of the remaining axioms.

Example 8: Define an evaluation τ by for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, for all weight function w and for all $i \in P$,

$$\tau^t_i(P, e, D^m) = 0.$$

Clearly, τ satisfies MCBCIY, but it violates 1WSFG and 2WSFG.

Example 9: Define an evaluation τ by for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, for all weight function for participants d and for all $i \in P$,

$$\tau^t_i(P, e, D^m) = \begin{cases} \Gamma^{f,t}_i(P, e, D^m) & \text{if } |P| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ satisfies 1WSFG, but it violates MCBCIY.

Example 10: Define an evaluation τ by for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, for all weight function for levels w and for all $i \in P$,

$$\tau^t_i(P, e, D^m) = \begin{cases} \Gamma^{w,t}_i(P, e, D^m) & \text{if } |P| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ satisfies 2WSFG, but it violates MCBCIY.

VI. Conclusions

In numerous situations, each participant is granted the flexibility to engage with an infinite range of energy levels (or make decisions and strategies). Participants are increasingly required to efficiently address multiple objectives in relative operational processes. Consequently, we concurrently address fuzzy behavior and multiple criteria situations. Weights naturally play a role in the framework of damage evaluation. For instance, when evaluating damage among investment projects, weights could be linked to the profitability of each project. Thus, we also explore generalized concepts for weighted evaluating. Differing from previous investigations on traditional transferable-utility situations and fuzzy transferable-utility situations, this paper presents several novel contributions.

- Simultaneously considering fuzzy behavior and multiple criteria situations, we consider the framework of multiple criteria fuzzy transferable-utility situations.
- By applying infinitesimal marginal damages under fuzzy behavior and multiple criteria situations simultaneously, we propose the IEENSD, the normalized marginal-index, and related axiomatic processes.
- To mitigate discrimination and relative effects caused by participants and its energy levels, we introduce two weighted extensions of the IEENSD and related axiomatic processes.
- All evaluations and related results are initially presented within the frameworks of traditional transferable-utility situations and fuzzy transferable-utility situations.

Building upon the key findings of this study, an intriguing prospect is to explore the extension of traditional evaluations by simultaneously incorporating infinitesimal marginal damages with the context of multiple criteria situations under fuzzy behavior. This endeavor is left for further investigation by interested readers.

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