Estimation of Multicomponent Stress-Strength Reliability with Exponentiated Generalized Inverse Rayleigh Distribution

Neama Salah Youssef Temraz

Abstract— **In this paper, an estimation of the multicomponent stress-strength reliability is introduced subject to the exponentiated generalized inverse Rayleigh distribution. Different methods of estimation are introduced to estimate the multicomponent stress-strength reliability. Simulation method is introduced to illustrate the steps of finding the estimates of the multicomponent stress-strength reliability. Asymptotic and bootstrap confidence intervals are proposed in order to find interval estimations for the multicomponent stress-strength reliability. A Bayesian estimation method is introduced for the multicomponent stress-strength reliability. A simulation study is introduced to obtain the estimates of the multicomponent stress-strength reliability for the different methods of estimation. A real data application is introduced to show how the exponentiated generalized inverse Rayleigh distribution is used to fit the real data sets.**

Index Terms— **Reliability, stress-strength, exponentiated generalized inverse Rayleigh distribution (EGIR), maximum likelihood estimation, least square estimation, Cramér-Von-Mises estimation, Bayesian estimation.**

I. Introduction

Bhattacharyya and Johnson [1] introduced an estimation of reliability in a multicomponent stress-strength model. Norman and Guttman [2] presented a Bayesian analysis of reliability in multicomponent stress-strength models. Rao and Kantam [3] presented an estimation of reliability in multicomponent stress strength model subject to log-logistic distribution. Rao [4] introduced an estimation of reliability in multicomponent stress-strength based on generalized exponential distribution. Rao et al. [5] proposed an estimation of reliability in multicomponent stress–strength based on two parameter exponentiated Weibull distribution. Dey et al. [6] Estimation of reliability of multicomponent stress strength for a Kumaraswamy distribution. Kızılaslan and Nadar [7] presented an estimation of the reliability in a multicomponent stress strength model based on a bivariate Kumaraswamy distribution. Hassan and Alohali [8] introduced an estimation of reliability in a multicomponent stress- strength model based on generalized linear failure rate distribution.

Kızılaslan [9] presented classical and Bayesian estimation of reliability in a multicomponent stress–strength model based on a general class of inverse exponentiated distributions.

Manuscript received November 11, 2023; revised May 21, 2024.
Neama Salah Youssef Temraz is an associate professor at Prince Sattam Bin Abdulaziz University, College of Sciences and Humanities, Saudi Arabia and an associate professor at Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt (e-mail: n.temraz@psau.edu.sa, sci_neama55@yahoo.com)

Kohansal [10] proposed an estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample. Akgül [11] presented an estimation for the reliability in multicomponent stress–strength model for Topp-Leone distribution. Rao et al. [12] introduced an estimation of reliability in a multicomponent stress–strength system for the exponentiated moment-based exponential distribution.

Jha et al. [13] presented an estimation of the multicomponent reliability by assuming the unit-Gompertz distribution. Hassan et al. [14] introduced an estimation of multicomponent stress-strength reliability following Weibull distribution based on upper record values. Alotaibi et al. [15] introduced Bayesian and non-Bayesian reliability estimation of multicomponent stress strength model for unit Weibull distribution. Kotb and Raqab [16] presented an estimation study of the reliability of multicomponent stress strength model based on modified Weibull distribution. Jia et al. [17] presented an inference on stress-strength reliability from inverse Weibull distribution based on first-failure progressively unified hybrid censored scheme. Jana and Bera [18] considered an interval estimation of stress strength reliability of k-out-of-n system when the stress and strength components follow inverse Weibull distributions. Ahmad et al. [19] presented a comparative inference on reliability estimation for a multi-component stress-strength model under power Lomax distribution with applications. Lio et al. [20] introduced an inference of the multicomponent stress strength reliability for Burr XII distribution. Pasha-Zanoosi et al. [21] presented multicomponent stress strength reliability with Exponentiated Teissier distribution. Zhang et al. [22] introduced a Bayesian inference of system reliability for multicomponent stress strength model under Marshall-Olkin Weibull distribution.

In this paper, the multicomponent stress-strength reliability model with the exponentiated generalized inverse Rayleigh distribution is introduced. Different methods of estimation for the multicomponent stress-strength reliability are discussed. The asymptotic, parametric bootstrap sampling and student̓ s t bootstrap sampling confidence intervals for the multicomponent stress-strength reliability are introduced. Bayesian estimation method and the credible interval for the multicomponent stress-strength reliability function in case of known and unknown parameters are presented. A simulation study is introduced to show the results for the different methods of estimation for the multicomponent stress-strength reliability. A real data application is introduced to show the results for the multicomponent stress-strength model.

II. EXPONENTIATED GENERALIZED INVERSE RAYLEIGH DISTRIBUTION

Fatima et al. [23] introduced the exponentiated generalized inverse Rayleigh distribution with cumulative distribution function and probability density function defined as follows:

$$F(x) = \left[1 - \left(1 - e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma}, x > 0, \gamma > 0, \alpha > 0, \lambda > 0$$

and

$$f(x) = \frac{2\alpha\gamma}{\lambda^2 x^3} e^{-(\lambda x)^{-2}} \left(1 - e^{-(\lambda x)^{-2}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1}$$

III. MULTICOMPONENT STRESS-STRENGTH RELIABILITY

Let X and Y are two independent random variables follow the exponentiated generalized inverse Rayleigh distribution, then the stress-strength reliability function will be given by:

$$R_{s,k} = P[at\ least\ s\ of\ the\ (X_1, X_2, \ldots, X_k)\ exceed\ Y]$$
$$= \sum_{i=s}^{k} \binom{k}{i} \int_0^{\infty} [1 - F(y)]^i [F(y)]^{k-i} dGy$$
$$= \sum_{i=s}^{k} \binom{k}{i} \int_0^{\infty} \left\{\left[1 - \left\{1 - \left(1 - e^{-(\lambda y)^{-2}}\right)^{\alpha}\right\}^{\gamma_1}\right]^i \left[1 - \left(1 - e^{-(\lambda y)^{-2}}\right)^{\alpha}\right]^{\gamma_1(k-i)} \frac{2\alpha\gamma_2}{\lambda^2 y^3} e^{-(\lambda y)^{-2}} \left(1 - e^{-(\lambda y)^{-2}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-(\lambda y)^{-2}}\right)^{\alpha}\right]^{\gamma_2-1}\right\} dGy$$

Let $t = \left(1 - e^{-(\lambda y)^{-2}}\right)$, then

$$R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{i} \binom{i}{j} \binom{k}{i} (-1)^j \alpha\gamma_2 \int_0^1 t^{\alpha-1} (1 - t^{\alpha})^{\gamma_1(j+k-i)+\gamma_2-1} dt$$

Let $r = (1 - t^{\alpha})$, then the multicomponent stress-strength reliability is obtained as:

$$R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{i} \binom{i}{j} \binom{k}{i} (-1)^j \frac{\gamma_2}{\gamma_1(j + k - i) + \gamma_2} \qquad (1)$$

IV. DIFFERENT METHODS OF ESTIMATION

Different methods of estimation will be discussed in order to find the estimators of the parameters of the exponentiated generalized inverse Rayleigh distribution and hence find the estimator of the multicomponent stress strength reliability function. The methods of estimation that will be discussed are the maximum likelihood, the least squares and Cramér-Von-Mises methods.

A. *Maximum Likelihood Estimation Method*

Let n system be put on a life testing experiment. Further assume that $X_{i1}, X_{i2}, \ldots, X_{ik}$ and $Y_i, i = 1,2, \ldots, n$ denote the observed data obtained using EGIR$(\alpha, \lambda, \gamma_1)$ and EGIR$(\alpha, \lambda, \gamma_2)$ distributions respectively. Thus, the likelihood function of these unknown parameters can be written as:

$$L(\alpha, \lambda, \gamma_1, \gamma_2; x, y) = \prod_{i=1}^{n} \left(\prod_{j=1}^{k} f(x_{ij})\right) g(y_i)$$
$$= \prod_{i=1}^{n} \prod_{j=1}^{k} \left\{\frac{2\alpha\gamma_1}{\lambda^2 {x_{ij}}^3} e^{-(\lambda x_{ij})^{-2}} \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1-1}\right\} \prod_{i=1}^{n} \left\{\frac{2\alpha\gamma_2}{\lambda^2 {y_i}^3} e^{-(\lambda y_i)^{-2}} \left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2-1}\right\}$$

The log-likelihood function is obtained as follows:

$$logL = n(k+1)\log(2) + n(k+1)\log(\alpha) + nk\log(\gamma_1) + n\log(\gamma_2) - 2n(k+1)\,log(\lambda) - 3\sum_{i=1}^{n}\sum_{j=1}^{k}\log(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{k}(\lambda x_{ij})^{-2} + (\alpha-1)\sum_{i=1}^{n}\sum_{j=1}^{k}\log\left(1 - e^{-(\lambda x_{ij})^{-2}}\right) + (\gamma_1 - 1)\sum_{i=1}^{n}\sum_{j=1}^{k}\log\left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right] - 3\sum_{i=1}^{n}\log(y_i) - \sum_{i=1}^{n}(\lambda y_i)^{-2} + (\alpha-1)\sum_{i=1}^{n}\log\left(1 - e^{-(\lambda y_i)^{-2}}\right) + (\gamma_2 - 1)\sum_{i=1}^{n}\log\left[1 - \left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]$$

The partial derivatives of the log-likelihood function with respect to $\alpha, \lambda, \gamma_1$ *and* γ_2 are obtained as follows:

$$\frac{\partial logL}{\partial \alpha} = \frac{n(k+1)}{\alpha} + \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left(1 - e^{-(\lambda x_{ij})^{-2}}\right) - (\gamma_1 - 1)\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{\left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\log\left(1 - e^{-(\lambda x_{ij})^{-2}}\right)}{\left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]} + \sum_{i=1}^{n}\log\left(1 - e^{-(\lambda y_i)^{-2}}\right) - (\gamma_2 - 1)\sum_{i=1}^{n}\frac{\left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\log\left(1 - e^{-(\lambda y_i)^{-2}}\right)}{\left[1 - \left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]}$$

$$\frac{\partial logL}{\partial \gamma_1} = \frac{nk}{\gamma_1} + \sum_{i=1}^{n}\sum_{j=1}^{k} log\left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]$$

$$\frac{\partial logL}{\partial \gamma_2} = \frac{n}{\gamma_2} + \sum_{i=1}^{n} log\left[1 - \left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]$$

$$\frac{\partial logL}{\partial \lambda} = \frac{-2n(k+1)}{\lambda} + 2\left(\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij} + \sum_{i=1}^{n} y_i\right)$$

$$-2(\alpha-1)\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{e^{-(\lambda x_{ij})}\left(\lambda x_{ij}\right)^{-3}x_{ij}}{1-e^{-(\lambda x_{ij})^{-2}}}$$
$$+2\alpha(\gamma_1-1)\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha-1}e^{-(\lambda x_{ij})^{-2}}\left(\lambda x_{ij}\right)^{-3}x_{ij}}{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]}$$
$$-2(\alpha-1)\sum_{i=1}^{n}\frac{e^{-(\lambda y_i)^{-2}}(\lambda y_i)^{-3}y_i}{1-e^{-(\lambda y_i)^{-2}}}$$
$$+2\alpha(\gamma_2-1)\sum_{i=1}^{n}\frac{\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha-1}e^{-(\lambda y_i)^{-2}}(\lambda y_i)^{-3}y_i}{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]}$$

Equating the partial derivatives to zero as follows:

$$\frac{\partial logL}{\partial\alpha}=0,\frac{\partial logL}{\partial\gamma_1}=0,\frac{\partial logL}{\partial\gamma_2}=0,\frac{\partial logL}{\partial\lambda}=0$$

And then solving the equations numerically yields the maximum likelihood estimators for the parameters $(\alpha,\gamma_1,\gamma_2,\lambda)$ denoted by $\hat{\alpha}^M,\hat{\gamma}_1^M,\hat{\gamma}_2^M$ and $\hat{\lambda}^M$.

The maximum likelihood estimators for the parameters γ_1 and γ_2 are obtained as:

$$\hat{\gamma}_1^M=\frac{-nk}{\sum_{i=1}^{n}\sum_{j=1}^{k}log\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]}$$
$$\hat{\gamma}_2^M=\frac{-n}{\sum_{i=1}^{n}log\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]}$$

The maximum likelihood estimator of the multicomponent stress-strength reliability will be given by:

$$\hat{R}_{s,k}^M=\sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^j\frac{\hat{\gamma}_2^M}{\hat{\gamma}_1^M(j+k-i)+\hat{\gamma}_2^M}\qquad(2)$$

B. Least Squares Estimation Method

The least squares estimators of the multicomponent stress strength parameters which dented by $\hat{\alpha}^L,\hat{\gamma}_1^L,\hat{\gamma}_2^L$ and $\hat{\lambda}^L$ can be obtained by minimizing the following function:

$$LS=\sum_{l=1}^{nk}\left[F(x_l)-\frac{l}{nk+1}\right]^2+\sum_{i=1}^{n}\left[F(y_i)-\frac{i}{n+1}\right]^2$$
$$=\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{l}{nk+1}\right\}^2$$
$$+\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{i}{n+1}\right\}^2$$

Or, equivalently solving the following differential equation after equating them to zero.

$$\frac{\partial LS}{\partial\alpha}=-2\gamma_1\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{l}{nk+1}\right\}$$
$$\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1-1}\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}$$
$$\log\left(1-e^{-(\lambda x_{ij})^{-2}}\right)$$
$$-2\gamma_2\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{i}{n+1}\right\}$$
$$\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2-1}\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\log\left(1-e^{-(\lambda y_i)^{-2}}\right)$$

$$\frac{\partial LS}{\partial\gamma_1}=2\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{l}{nk+1}\right\}$$
$$\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}\log\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]$$

$$\frac{\partial LS}{\partial\gamma_2}=2\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{i}{n+1}\right\}$$
$$\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}\log\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]$$

$$\frac{\partial LS}{\partial\lambda}=4\gamma_1\alpha\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{l}{nk+1}\right\}$$
$$\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1-1}\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha-1}$$
$$e^{-(\lambda x_{ij})^{-2}}\left(\lambda x_{ij}\right)^{-3}x_{ij}$$
$$+4\gamma_2\alpha\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{i}{n+1}\right\}$$
$$\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2-1}\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha-1}$$
$$e^{-(\lambda y_i)^{-2}}(\lambda y_i)^{-3}y_i$$

The least squares estimator of the multicomponent stress-strength reliability can be obtained by:

$$\hat{R}_{s,k}^L=\sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^j\frac{\hat{\gamma}_2^L}{\hat{\gamma}_1^L(j+k-i)+\hat{\gamma}_2^L}\qquad(3)$$

C. Cramér-Von-Mises Estimation Method

The Cramér-Von-Mises estimators of the multicomponent stress strength parameters which denoted by $\hat{\alpha}^C,\hat{\gamma}_1^C,\hat{\gamma}_2^C$ and $\hat{\lambda}^C$ can be obtained by minimizing the following function:

$$CM=\frac{1}{12nk}+\sum_{l=1}^{nk}\left[F(x_l)-\frac{2l-1}{2nk}\right]^2+\frac{1}{12n}$$
$$+\sum_{i=1}^{n}\left[F(y_i)-\frac{2i}{2n+1}\right]^2$$
$$=\frac{1}{12nk}+\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{2l}{2nk+1}\right\}^2$$
$$+\frac{1}{12n}+\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{2i}{2n+1}\right\}^2$$

Or, equivalently solving the following differential equation after equating them to zero.

$$\frac{\partial LS}{\partial\alpha}=-2\gamma_1\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{2l}{2nk+1}\right\}$$
$$\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1-1}\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\log\left(1-e^{-(\lambda x_{ij})^{-2}}\right)$$
$$-2\gamma_2\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{2i}{2n+1}\right\}$$

$$\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2-1}\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\log\left(1-e^{-(\lambda y_i)^{-2}}\right)$$

$$\frac{\partial LS}{\partial \gamma_1}=2\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{2l}{2nk+1}\right\}\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}\log\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]$$

$$\frac{\partial LS}{\partial \gamma_2}=2\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{2i}{2n+1}\right\}\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}\log\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]$$

$$\frac{\partial LS}{\partial \lambda}=4\gamma_1\alpha\sum_{l=1}^{nk}\left\{\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1}-\frac{2l}{2nk+1}\right\}\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]^{\gamma_1-1}\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha-1}e^{-(\lambda x_{ij})^{-2}}(\lambda x_{ij})^{-3}x_{ij}$$
$$+4\gamma_2\alpha\sum_{i=1}^{n}\left\{\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2}-\frac{2i}{2n+1}\right\}\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]^{\gamma_2-1}\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha-1}e^{-(\lambda y_i)^{-2}}(\lambda y_i)^{-3}y_i$$

The Cramér-Von-Mises estimator of the multicomponent stress-strength reliability can be obtained by:

$$\hat{R}_{s,k}^{C}=\sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^{j}\frac{\hat{\gamma}_2^{C}}{\hat{\gamma}_1^{C}(j+k-i)+\hat{\gamma}_2^{C}} \quad (4)$$

V. Monte Carlo Simulation Algorithm

Step 1: Set initial values of the parameters $(\alpha,\lambda,\gamma_1,\gamma_2)$.
Step 2: Choose (s, k) and sample size n.
Step 3: Using the inversion method to generate random samples from random variables X_i and Y_j at the initial values of the parameters $(\alpha,\lambda,\gamma_1,\gamma_2)$by applying the inversion method as follows:

$$X_{ij}=\frac{1}{\lambda}\left[-\log\left(1-\left(1-u_{ij}^{\frac{1}{\gamma_1}}\right)^{\alpha}\right)\right]^{-\frac{1}{2}},\quad 0<u_{ij}<1, i=1,2,\dots,n, j=1,2,\dots k$$

$$Y_i=\frac{1}{\lambda}\left[-\log\left(1-\left(1-s_i^{\frac{1}{\gamma_2}}\right)^{\alpha}\right)\right]^{-\frac{1}{2}},\quad 0<s_i<1, i=1,2,\dots,n$$

Step 4: Using the method of Newton-Raphson to obtain the estimates of the parameters according to the different methods of estimation.
Step 5: The estimates of the multicomponent stress-strength reliability can be obtained by substituting in Equations (2), (3) and (4).
Step 6: Repeat steps from 3 to 5, L times. The mean squared error (MSE) is given by

$$MSE=\frac{\sum_{i=1}^{L}\left(\hat{R}_{s,k,i}-R_{s,k}\right)^2}{L}$$

VI. Asymptotic Confidence Interval

The observed Fisher information matrix of the parameters $\alpha,\lambda,\gamma_1,\gamma_2$ is given by:

$$I(\theta)=\begin{pmatrix}-\frac{\partial^2 logL}{\partial\alpha^2} & -\frac{\partial^2 logL}{\partial\alpha\partial\gamma_1} & -\frac{\partial^2 logL}{\partial\alpha\partial\gamma_2} & -\frac{\partial^2 logL}{\partial\alpha\partial\lambda}\\ -\frac{\partial^2 logL}{\partial\gamma_1\partial\alpha} & -\frac{\partial^2 logL}{\partial{\gamma_1}^2} & -\frac{\partial^2 logL}{\partial\gamma_1\partial\gamma_2} & -\frac{\partial^2 logL}{\partial\gamma_1\partial\lambda}\\ -\frac{\partial^2 logL}{\partial\gamma_2\partial\alpha} & -\frac{\partial^2 logL}{\partial\gamma_2\partial\gamma_1} & -\frac{\partial^2 logL}{\partial{\gamma_2}^2} & -\frac{\partial^2 logL}{\partial\gamma_2\partial\lambda}\\ -\frac{\partial^2 logL}{\partial\lambda\partial\alpha} & -\frac{\partial^2 logL}{\partial\lambda\partial\gamma_1} & -\frac{\partial^2 logL}{\partial\lambda\partial\gamma_2} & -\frac{\partial^2 logL}{\partial\lambda^2}\end{pmatrix}$$

where

$$\frac{\partial^2 logL}{\partial\gamma_2\partial\gamma_1}=0 \; and \; \frac{\partial^2 logL}{\partial\gamma_1\partial\gamma_2}=0$$

The asymptotic variance of an estimate $\hat{R}_{s,k}$ will be given by:

$$V(\hat{R}_{s,k})=\left(-\frac{\partial^2 logL}{\partial{\hat{\gamma}_1}^2}\right)^{-1}\left(\frac{\partial\hat{R}_{s,k}}{\partial\hat{\gamma}_1}\right)^2+\left(-\frac{\partial^2 logL}{\partial{\hat{\gamma}_2}^2}\right)^{-1}\left(\frac{\partial\hat{R}_{s,k}}{\partial\hat{\gamma}_2}\right)^2$$

$$V(\hat{R}_{s,k})=\frac{\hat{\gamma}_1^2}{nk}\left\{\sum_{i=1}^{n}\sum_{j=1}^{k}\binom{k}{i}(-1)^{j+1}\frac{\hat{\gamma}_2(j+k-i)}{[\hat{\gamma}_1(j+k-i)+\hat{\gamma}_2]^2}\right\}^2+\frac{\hat{\gamma}_2^2}{n}\left\{\sum_{i=1}^{n}\sum_{j=1}^{k}\binom{k}{i}(-1)^{j}\frac{\hat{\gamma}_1(j+k-i)}{[\hat{\gamma}_1(j+k-i)+\hat{\gamma}_2]^2}\right\}^2$$

An asymptotic $100(1-\delta)\%$ confidence interval of $R_{s,k}$ will be given by:

$$\left[\hat{R}_{s,k}\pm z_{1-\frac{\delta}{2}}\sqrt{V(\hat{R}_{s,k})}\right]$$

VII. Bootstrap Confidence Interval for $R_{s,k}$

A. *Parametric Bootstrap Sampling*

Step1: From given samples $(x_{i1},\dots,x_{ik})$ and $(y_1,\dots,y_n)$, compute the estimates $(\hat{\alpha},\hat{\gamma}_1,\hat{\gamma}_2,\hat{\lambda})$ of $(\alpha,\gamma_1,\gamma_2,\lambda)$.
Step 2: Generate a bootstrap sample of size *nk* $(x_{i1}^*,\dots,x_{in}^*), i=1,\dots,n$ from EGIR$(\hat{\alpha},\hat{\lambda},\hat{\gamma}_1)$ and generate a bootstrap sample $(y_1^*,\dots,y_n^*)$ of size n from EGIR$(\hat{\alpha},\hat{\lambda},\hat{\gamma}_2)$
Step 3: Compute the estimates $(\hat{\alpha}^*,\hat{\gamma}_1^*,\hat{\gamma}_2^*,\hat{\lambda}^*)$ of $(\alpha,\gamma_1,\gamma_2,\lambda)$ and then compute the bootstrap estimates $\hat{R}_{s,k}^*$ of $R_{s,k}$
Step 4: Repeat Step 2 and 3, B times to obtain a set of bootstrap samples of R, say $\left\{\hat{R}_{s,k}^{*(j)}, j=1,\dots,B\right\}$
Step 5: Order $\hat{R}_{s,k}^{*(j)}, j=1,\dots,B$, such that $\hat{R}_{s,k}^{*(1)}<\cdots<\hat{R}_{s,k}^{*(B)}$, a $100(1-\delta)\%$ bootstrap confidence interval will be given by:

$$\left[\hat{R}_{s,k,B(\alpha/2)}^*,\hat{R}_{s,k,B(1-\alpha/2)}^*\right]$$

B. *Student's t Bootstrap Sampling*

Step1: From given samples $(x_{i1},\dots,x_{ik})$ and $(y_1,\dots,y_n)$, compute the estimates $(\hat{\alpha},\hat{\gamma}_1,\hat{\gamma}_2,\hat{\lambda})$ of $(\alpha,\gamma_1,\gamma_2,\lambda)$ and compute the estimate $\hat{R}_{s,k}$
Step 2: Generate a bootstrap sample of size nk $(x_{i1}^*,\dots,x_{in}^*), i=1,\dots,n$ from EGIR$(\hat{\alpha},\hat{\lambda},\hat{\gamma}_1)$ and generate a bootstrap sample $(y_1^*,\dots,y_n^*)$ of size n from EGIR$(\hat{\alpha},\hat{\lambda},\hat{\gamma}_2)$
Step 3: Compute the estimates $(\hat{\alpha}^*,\hat{\gamma}_1^*,\hat{\gamma}_2^*,\hat{\lambda}^*)$ of $(\alpha,\gamma_1,\gamma_2,\lambda)$ and then compute the bootstrap estimates $\hat{R}_{s,k}^*$ of $R_{s,k}$
Step 4: Repeat Step 2 and 3, B times to obtain a set of bootstrap samples of R, say $\left\{\hat{R}_{s,k}^{*(j)}, j=1,\dots,B\right\}$
Step 5: Order $\hat{R}_{s,k}^{*(j)}, j=1,\dots,B$, such that $\hat{R}_{s,k}^{*(1)}<\cdots<\hat{R}_{s,k}^{*(B)}$

Step 6: Compute the sample standard deviation of $\left\{\hat{R}^{*(j)}_{s,k}, j = 1, \dots, B\right\}$ such that:

$$sd\left(\hat{R}^{*}_{s,k}\right) = \sqrt{V\left(\hat{R}^{*}_{s,k}\right)} = \sqrt{\frac{1}{B}\sum_{j=1}^{B}\left(\hat{R}^{*(j)}_{s,k} - \hat{R}_{s,k}\right)^2}$$

And compute the t^* statistic such that:

$$t^{*(j)} = \frac{\hat{R}^{*(j)}_{s,k} - \hat{R}_{s,k}}{sd\left(\hat{R}^{*}_{s,k}\right)}$$

A $100(1-\delta)\%$ bootstrap confidence interval will be given by:

$$\hat{R}_{s,k} \pm t^{*B(\alpha/2)} sd\left(\hat{R}^{*}_{s,k}\right)$$

VIII. Bayesian Estimation Method

A. Unknown Parameters

The Bayesian estimator for the stress-strength reliability will be obtained assuming that the parameters $\alpha, \gamma_1, \gamma_2\ and\ \lambda$ are independent random variables with priors follow gamma distribution as follows:

$$\alpha \sim Gamma(a_1, b_1)$$
$$\gamma_1 \sim Gamma(a_2, b_2)$$
$$\gamma_2 \sim Gamma(a_3, b_3)$$
$$\lambda \sim Gamma(a_4, b_4)$$

The joint prior density function of the parameters $\alpha, \gamma_1, \gamma_2\ and\ \lambda$ is given by:

$$\pi(\alpha, \gamma_1, \gamma_2, \lambda) \propto \alpha^{a_1-1}\gamma_1^{a_2-1}\gamma_2^{a_3-1}\lambda^{a_4-1}e^{-b_1\alpha - b_2\gamma_1 - b_3\gamma_2 - b_4\lambda}$$

The joint posterior density function of $\alpha, \gamma_1, \gamma_2\ and\ \lambda$ given the data (x, y) is given by:

$$\pi(\alpha, \gamma_1, \gamma_2, \lambda | x, y) \propto \pi(\alpha, \gamma_1, \gamma_2, \lambda) L(\alpha, \gamma_1, \gamma_2, \lambda | x, y)$$

$$\pi(\alpha, \gamma_1, \gamma_2, \lambda | x, y) \propto \alpha^{n(k+1)+a_1-1}\gamma_1^{nk+a_2-1}\gamma_2^{n+a_3-1}\lambda^{-2n(k+1)+a_4-1} e^{-\varphi_1\alpha - \varphi_2\gamma_1 - \varphi_3\gamma_2 - b_4\lambda - \varphi_4}$$

where

$$\varphi_1 = b_1 - \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left(1 - e^{-(\lambda x_{ij})^{-2}}\right) - \sum_{i=1}^{n}\log\left(1 - e^{-(\lambda y_i)^{-2}}\right)$$

$$\varphi_2 = b_2 - \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]$$

$$\varphi_3 = b_3 - \sum_{i=1}^{n}\log\left[1 - \left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]$$

$$\varphi_4 = \sum_{i=1}^{n}\sum_{j=1}^{k}(\lambda x_{ij})^{-2} + \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left(1 - e^{-(\lambda x_{ij})^{-2}}\right) + \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right] + \sum_{i=1}^{n}(\lambda y_i)^{-2} + \sum_{i=1}^{n}\log\left(1 - e^{-(\lambda y_i)^{-2}}\right) + \sum_{i=1}^{n}\log\left[1 - \left(1 - e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]$$

The marginal posterior distributions of $\alpha, \gamma_1, \gamma_2\ and\ \lambda$ can be deduced as:

$$\pi_1^*(\alpha | \lambda, x, y) \propto Gamma(n(k+1) + a_1, \varphi_1)$$
$$\pi_2^*(\gamma_1 | \lambda, \alpha, x) \propto Gamma(nk + a_2, \varphi_2)$$
$$\pi_3^*(\gamma_2 | \lambda, \alpha, y) \propto Gamma(n + a_3, \varphi_3)$$
$$\pi_4^*(\lambda | \alpha, x, y) \propto \lambda^{-2n(k+1)+a_4-1}e^{-b_4\lambda - \varphi_4}$$

The posterior distributions of the parameters $\alpha, \gamma_1 and\ \gamma_2$ are gamma distributions and the posterior distribution of λ is unknown. Metropolis-Hastings (MH) algorithm can be applied to simulate random samples from the posterior density of λ. The Markov Chain Monte Carlo (MCMC) simulation method will be applied to obtain the Bayesian estimation of the multicomponent stress-strength reliability.

Step1: Choose initial values $\alpha^0, \gamma_1^0, \gamma_2^0, \lambda^0$
Step 2: Set $i = 1$
Step 3: Generate $\alpha^{(i)}$from $Gamma(n(k+1) + a_1, \varphi_1)$
Step 4: Generate $\gamma_1^{(i)}$from $Gamma(nk + a_2, \varphi_2)$
Step 5: Generate $\gamma_2^{(i)}$from $Gamma(n + a_3, \varphi_3)$
Step 6: Generate $\lambda^{(i)}$from $\pi_4^*(\lambda | \alpha, x, y)$ using MH algorithm with proposals generated from the normal distribution $N\left(\lambda^{(i-1)}, V(\lambda^{(i-1)})\right)$ where $V(\lambda) = \left(-\frac{\partial^2 logL}{\partial\lambda^2}\right)^{-1}$
Step 7: Compute $R_{s,k}^{(i)}$ from Equation (1)
Step 8: Set $i = i + 1$
Step 9: Repeat steps from 3 to 9, T times. The Bayes estimator of the multicomponent stress-strength reliability under the squared error loss will be given as:

$$\hat{R}^{B}_{s,k} = \frac{1}{T}\sum_{t=1}^{T} R_{s,k}^{(t)}$$

To construct the credible interval for R, Firstly, reorder $R^{(t)}$ such that $R_{s,k}^{(1)} < \dots < R_{s,k}^{(t)}$. Then a $100(1-\varepsilon)\%$ credible interval of $R^{B}_{s,k}$ becomes $[R_{s,k}^{(T\varepsilon/2)}, R_{s,k}^{(T(1-\varepsilon/2))}]$.

B. known Parameters

The parameters $\alpha\ and\ \lambda$ are assumed to be known and the parameters $\gamma_1\ and\ \gamma_2$ are assumed to have prior distributions follow gamma distribution. The joint prior distribution of $\gamma_1\ and\ \gamma_2$ will be given by:

$$\pi(\gamma_1, \gamma_2) = \frac{\gamma_1^{a_2-1}\gamma_2^{a_3-1}}{\Gamma(a_2)\Gamma(a_3)}e^{-(b_2\gamma_1 + b_3\gamma_2)},$$

$$\gamma_1 > 0, \gamma_2 > 0, a_2 > 0, a_3 > 0, b_2 > 0, b_3 > 0$$

The joint posterior distribution of $\gamma_1\ and\ \gamma_2$ will be given by:

$$\pi(\gamma_1, \gamma_2 | x, y) = \pi(\gamma_1, \gamma_2) L(x, y | \alpha, \gamma_1, \gamma_2, \lambda)$$

$$\pi(\gamma_1, \gamma_2 | x, y) = \frac{\pi(\gamma_1, \gamma_2) L(\alpha, \gamma_1, \gamma_2, \lambda | x, y)}{\int_0^{\infty}\int_0^{\infty}\pi(\gamma_1, \gamma_2) L(\alpha, \gamma_1, \gamma_2, \lambda | x, y)\, d\gamma_1 d\gamma_2}$$

$$\pi(\gamma_1, \gamma_2 | x, y) = \left\{b_2 - \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]\right\}^{a_2+nk} \frac{\gamma_1^{a_2+nk-1}}{\Gamma(a_2+nk)} e^{-\gamma_1\left\{b_2 - \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left[1 - \left(1 - e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right]\right\}}$$

$$\left\{b_3-\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]\right\}^{a_3+n}$$
$$\frac{\gamma_2^{a_3+n-1}}{\Gamma(a_3+n)}\,e^{-\gamma_2\left\{b_3-\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]\right\}}$$

The Bayes estimator of the multicomponent stress-strength reliability under the squared error loss will be given by:

$$\hat{R}_{s,k}^{B}=\int_0^\infty\int_0^\infty R_{s,k}\,\pi(\gamma_1,\gamma_2|x,y)d\gamma_1 d\gamma_2$$

$$\hat{R}_{s,k}^{B}=\sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^j\frac{w_1^{a_2+nk}w_2^{a_3+n}}{\Gamma(a_2+nk)\Gamma(a_3+n)}$$
$$\int_0^\infty\int_0^\infty\frac{\gamma_2}{\gamma_1(j+k-i)+\gamma_2}\gamma_1^{a_2+nk-1}\gamma_2^{a_3+n-1}\,e^{-\gamma_1 w_1}e^{-\gamma_2 w_2}d\gamma_1 d\gamma_2$$

Where

$$w_1=b_2-\sum_{i=1}^{n}\sum_{j=1}^{k}\log\left[1-\left(1-e^{-(\lambda x_{ij})^{-2}}\right)^{\alpha}\right],$$
$$w_2=b_3-\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda y_i)^{-2}}\right)^{\alpha}\right]$$

Let $z_1=\frac{\gamma_2}{\gamma_1(j+k-i)+\gamma_2}, z_2=\gamma_1(j+k-i)+\gamma_2$ where

$0<z_1<1, z_2>0$. And hence $\gamma_2=z_1z_2, \gamma_1=\frac{z_2(1-z_1)}{(j+k-i)}$ and the Jacobian is $\frac{-z_2}{(j+k-i)}$

$$\hat{R}_{s,k}^{B}=\sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^j\frac{w_1^{a_2+nk}w_2^{a_3+n}}{(j+k-i)^{a_2+nk}\Gamma(a_2+nk)\Gamma(a_3+n)}$$
$$\int_0^1\int_0^\infty z_1^{a_3+n}\,z_2^{a_2+a_3+n(k+1)-1}\,(1-z_1)^{a_2+nk-1}$$
$$e^{-z_2\left[\frac{w_1(1-z_1)}{(j+k-i)}+w_2z_1\right]}dz_1dz_2$$

$$\hat{R}_{s,k}^{B}=\sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^j\,\frac{(1-\rho)^{a_3+n}}{\beta(a_2+nk,a_3+n)}$$
$$\int_0^1 z_1^{a_3+n}\,(1-z_1)^{a_2+nk-1}(1-\rho z_1)^{-\tau}dz_1$$

Where

$$\tau=a_2+a_3+n(k+1), \rho=1-\frac{(j+k-i)w_2}{w_1}$$

The Bayes estimator of the multicomponent stress strength reliability is given by:

$$\hat{R}_{s,k}^{B}=$$
$$\begin{cases}\sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^j\frac{(1-\rho)^{a_3+n}(a_3+n)}{\tau}\,{}_2F_1(\tau,a_3+n+1,\tau+1,\rho), & if\ |\rho|<1\\ \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{i}{j}\binom{k}{i}(-1)^j\frac{(a_3+n)}{\tau(1-\rho)^{a_2+nk}}\,{}_2F_1\left(\tau,a_2+nk,\tau+1,\frac{\rho}{\rho-1}\right), & if\ \rho<-1\end{cases}$$

In order to find the credible interval for the multicomponent stress strength reliability, the following steps will be applied. It can be shown that from the posterior distributions of the parameters γ_1, γ_2 and the relations between gamma and chi-square distributions, the following relation are held:

$$2\varphi_2\gamma_1\sim\chi^2_{2(nk+a_2)}$$

and

$$2\varphi_3\gamma_2\sim\chi^2_{2(n+a_3)}$$

Then posterior distribution of R can be written as:

$$\frac{1}{1+\frac{(n+a_3)\varphi_2}{(nk+a_2)\varphi_3}\,F\big(2(n+a_3),2(nk+a_2)\big)}$$

And therefore a $100(1-\varepsilon)\%$ credible interval for R will be given by:

$$\left[\left\{1+\frac{(n+a_3)\varphi_2}{(nk+a_2)\varphi_3}\,F_{\frac{\varepsilon}{2},2(n+a_3),2(nk+a_2)}\right\}^{-1},\right.$$
$$\left.\left\{1+\frac{(n+a_3)\varphi_2}{(nk+a_2)\varphi_3}\,F_{1-\frac{\varepsilon}{2},2(n+a_3),2(nk+a_2)}\right\}^{-1}\right]$$

The Markov Chain Monte Carlo (MCMC) simulation method can also be applied in order to obtain a Bayesian estimator of the multicomponent stress-strength reliability as follows:

Step1: Choose initial values $\alpha^0, \gamma_1^0, \gamma_2^0, \lambda^0$
Step 2: Set $i=1$
Step 3: Generate $\gamma_1^{(i)}$ from $Gamma(nk+a_2,\varphi_2)$
Step 4: Generate $\gamma_2^{(i)}$ from $Gamma(n+a_3,\varphi_3)$
Step 5: Compute $R_{s,k}^{(i)}$ from Equation ()
Step 6: Set $i=i+1$
Step 7: Repeat steps from 3 to 9, T times. The Bayes estimator of the multicomponent stress-strength reliability under the squared error loss will be given as:

$$\hat{R}_{s,k}^{B}=\frac{1}{T}\sum_{t=1}^{T}R_{s,k}^{(t)}$$

IX. Simulation Results

Monte Carlo simulation is performed in two cases: (s, k) = (1, 3) and (2, 4) with (L = 1000). The samples are generated using initial values of the parameters from the exponentiated generalized inverse Rayleigh distribution. The samples are taken of different sizes: (10, 10), (30, 30), (50, 50) and (100, 100). The results for the estimates the multicomponent strength-stress reliability of different methods of estimation with the mean squared error are obtained in Table I. The 95% asymptotic, parametric bootstrap sampling and student̓ s t bootstrap sampling confidence intervals are obtained in Table II. The results for the Bayesian estimates and 95% credible intervals for the multicomponent strength-stress reliability for the two priors: prior-I ($\xi_i=0.5, \eta_i=0.5, i=1,\dots 4$) and prior-II ($\xi_i=2, \eta_i=3, i=1,\dots 4$) when the parameters are unknown and known are obtained in Tables III and IV.
From the results obtained in Tables I – IV, it can be observed that:

1. The mean squared errors decrease as the sample size increases in all methods of estimation.
2. The lengths of the asymptotic, parametric bootstrap sampling and student̓ s t bootstrap sampling

confidence intervals decrease as the sample size increases.

3. The lengths of the credible intervals decrease as the sample size increases.
4. The lengths of the asymptotic confidence intervals are smaller than the lengths of the parametric bootstrap sampling confidence intervals.
5. The lengths of the parametric bootstrap sampling confidence intervals are smaller than the lengths of the student's t bootstrap sampling confidence intervals.
6. The lengths of the credible intervals when the parameters are unknown is wider than other intervals.
7. The lengths of the credible intervals when the parameters are known is smaller than other intervals.

X. Real Data Application

The data below show the number of cycles to failure for a group of 60 electrical appliances in a life test reported by Lawless [24]. The failure times have been ordered and the results are:

14, 34, 59, 61, 69, 80, 123, 142, 165, 210, 381, 464, 479, 556, 574, 839, 917, 969, 991, 1064, 1088, 1091, 1174, 1270, 1275, 1355, 1397, 1477, 1578, 1649, 1702, 1893, 1932, 2001, 2161, 2292, 2326, 2337, 2628, 2785, 2811, 2886, 2993, 3122, 3248, 3715, 3790, 3857, 3912, 4100, 4106, 4116, 4315, 4510, 4584, 5267, 5299, 5583, 6065, 9701

This data can be divided as follows:

1. The first element in the data set is Y_1,
2. The second element in the data set to 10th element represent $X_{1j}, j = 1, \dots, 9$,
3. The 11th element represents Y_2,
4. The elements from 12th to 20th represent $X_{2j}, j = 1, \dots, 9$,
5. And so on to obtain the data for Y and X as follows:

$$Y = \begin{pmatrix} 14 \\ 381 \\ 1088 \\ 1702 \\ 2811 \\ 4106 \end{pmatrix},$$

TABLE I
Results of estimates of different methods at $\alpha = 0.5, \gamma_1 = 0.5, \gamma_2 = 0.5$ *and* $\lambda = 0.5$

(s, k)	n	True $R_{s,k}$	MLE		LSE		CVME	
			$\hat{R}^M_{s,k}$	MSE	$\hat{R}^L_{s,k}$	MSE	$\hat{R}^C_{s,k}$	MSE
(1, 3)	10	0.75	0.7149	0.0060	0.7544	0.0001	0.7339	0.0001
	30		0.7319	0.0014	0.7476	2.9666e-05	0.7836	2.8015e-05
	50		0.7895	0.0009	0.7459	2.0222e-05	0.7484	2.4131e-05
	100		0.7514	0.0004	0.7479	1.4292e-05	0.7503	1.9599e-05
(2, 4)	10	0.60	0.6219	0.0110	0.6020	0.0001	0.5981	0.0001
	30		0.4456	0.0031	0.5957	5.6311e-05	0.5984	6.3994e-05
	50		0.6276	0.0018	0.5987	5.0709e-05	0.5987	4.7424e-05
	100		0.6245	0.0008	0.6087	2.8043e-05	0.6087	3.1091e-05

TABLE II
Results for the 95% asymptotic, parametric bootstrap sampling and student's t bootstrap sampling confidence intervals (with lengths) at $\alpha = 0.5, \gamma_1 = 0.5, \gamma_2 = 0.5$ *and* $\lambda = 0.5$

(s, k)	n	Maximum Likelihood			Least Squares			Cramér-Von-Mises		
		A.C.I	p-Boot	t-Boot	A.C.I	p-Boot	t-Boot	A.C.I	p-Boot	t-Boot
(1, 3)	10	[0.5691, 0.8608] (0.2917)	[0.3636, 0.7438] (0.3802)	[0.3635, 0.8392] (0.4757)	[0.6218, 0.8870] (0.2652)	[0.5379, 0.9104] (0.3725)	[0.5379, 0.9709] (0.4330)	[0.5941, 0.8736] (0.2795)	[0.5149, 0.9328] (0.4179)	[0.5149, 0.9528] (0.4379)
	30	[0.6508, 0.8130] (0.1622)	[0.6104, 0.7968] (0.1864)	[0.6103, 0.8281] (0.2178)	[0.6697, 0.8256] (0.1559)	[0.6276, 0.8435] (0.2159)	[0.6276, 0.8650] (0.2374)	[0.7135, 0.8536] (0.1401)	[0.6266, 0.8410] (0.2144)	[0.6266, 0.9405] (0.3139)
	50	[0.7363, 0.8427] (0.1064)	[0.6708, 0.7922] (0.1214)	[0.6707, 0.8117] (0.1410)	[0.6852, 0.8065] (0.1213)	[0.6731, 0.8099] (0.1368)	[0.6731, 0.8186] (0.1455)	[0.6881, 0.8086] (0.1205)	[0.6657, 0.8160] (0.1503)	[0.6657, 0.8310] (0.1653)
	100	[0.7092, 0.7937] (0.0845)	[0.7041, 0.7888] (0.0847)	[0.7040, 0.7987] (0.0947)	[0.7052, 0.7905] (0.0853)	[0.6985, 0.7977] (0.0992)	[0.6986, 0.8102] (0.1116)	[0.7079, 0.7927] (0.0848)	[0.7020, 0.7970] (0.0950)	[0.7019, 0.7987] (0.0968)
(2, 4)	10	[0.4400, 0.8038] (0.3638)	[0.3794, 0.7769] (0.3975)	[0.3794, 0.8644] (0.4850)	[0.4154, 0.7886] (0.3732)	[0.2786, 0.8478] (0.5692)	[0.2786, 0.9254] (0.6468)	[0.4106, 0.7856] (0.3750)	[0.2481, 0.8743] (0.6262)	[0.2480, 0.9481] (0.7001)
	30	[0.3273, 0.5639] (0.2366)	[0.3204, 0.5582] (0.2378)	[0.3204, 0.5708] (0.2504)	[0.4871, 0.7043] (0.2172)	[0.4980, 0.7116] (0.2136)	[0.4981, 0.9983] (0.5002)	[0.4902, 0.7067] (0.2165)	[0.4647 0.7264] (0.2617)	[0.4646, 0.7322] (0.2676)
	50	[0.5469, 0.7082] (0.1613)	[0.5306, 0.7014] (0.1708)	[0.5306, 0.7246] (0.1940)	[0.5150, 0.6825] (0.1675)	[0.5090, 0.6824] (0.1734)	[0.5090, 0.7083] (0.1993)	[0.5150, 0.6825] (0.1675)	[0.4942, 0.6942] (0.2000)	[0.4941, 0.7033] (0.2092)
	100	[0.5672, 0.6818] (0.1146)	[0.5583, 0.6760] (0.1177)	[0.5582, 0.6907] (0.1325)	[0.5501, 0.6672] (0.1171)	[0.5119, 0.6800] (0.1681)	[0.5118, 0.7055] (0.1937)	[0.5501, 0.6672] (0.1171)	[0.4966, 0.6791] (0.1825)	[0.4966, 0.7208] (0.2242)

$$X = \begin{pmatrix} 34 & 59 & 61 & 69 & 80 & 123 & 142 & 165 & 210 \\ 464 & 479 & 556 & 574 & 839 & 917 & 969 & 991 & 1064 \\ 1091 & 1174 & 1270 & 1275 & 1355 & 1397 & 1477 & 1578 & 1649 \\ 1893 & 1932 & 2001 & 2161 & 2292 & 2326 & 2337 & 2628 & 2785 \\ 2886 & 2993 & 3122 & 3248 & 3715 & 3790 & 3857 & 3912 & 4100 \\ 4116 & 4315 & 4510 & 4584 & 2567 & 5299 & 5583 & 6065 & 9701 \end{pmatrix}$$

The results for the maximum likelihood, least squares, Cramér-Von-Mises and Bayesian estimates are obtained in Table V when k = 5.

TABLE III
Results for Bayesian estimates and the credible intervals with lengths at $\alpha = 0.5, \gamma_1 = 0.5, \gamma_2 = 0.5$ *and* $\lambda = 0.5$ when the parameters are unknown

(s, k)	n	Prior-I			Prior-II		
		$\hat{R}_B$	MSE	C.I.	$\hat{R}_B$	MSE	C.I.
(1, 3)	10	0.7285	0.0659	[0.1528, 0.9996] (0.8468)	0.7380	0.0366	[0.2526, 0.9772] (0.7245)
	30	0.7331	0.0462	[0.2184, 0.9973] (0.7789)	0.7419	0.0309	[0.3153, 0.9722] (0.6568)
	50	0.7373	0.0357	[0.2815, 0.9919] (0.7103)	0.7472	0.0278	[0.3363, 0.9714] (0.6350)
	100	0.7362	0.0280	[0.3618, 0.9823] (0.6204)	0.7484	0.0213	[0.4152, 0.9572] (0.5420)
(2, 4)	10	0.6296	0.1109	[0.0264, 0.9992] (0.9728)	0.6181	0.0835	[0.0829, 0.9524] (0.8695)
	30	0.6328	0.0893	[0.0844, 0.9974] (0.9130)	0.6247	0.0717	[0.1185, 0.9594] (0.8408)
	50	0.6335	0.0754	[0.1150, 0.9924] (0.8773)	0.6294	0.0616	[0.1625, 0.9523] (0.7898)
	100	0.6349	0.0610	[0.1921, 0.9835] (0.7914)	0.6348	0.0526	[0.2071, 0.9483] (0.7412)

TABLE IV
Results for Bayesian estimates and the credible intervals with lengths at $\alpha = 0.5, \gamma_1 = 0.5, \gamma_2 = 0.5$ *and* $\lambda = 0.5$ when the parameters are known

(s, k)	n	Prior-I			Prior-II		
		$\hat{R}_B$	MSE	C.I.	$\hat{R}_B$	MSE	C.I.
(1, 3)	10	0.8942	0.0225	[0.8061, 0.9630] (0.1568)	0.8899	0.0222	[0.7883, 0.9552] (0.1669)
	30	0.8977	0.0224	[0.8454, 0.9374] (0.0919)	0.8975	0.0222	[0.8523, 0.9334] (0.0811)
	50	0.8983	0.0223	[0.8609, 0.9299] (0.0689)	0.8980	0.0222	[0.8616, 0.9295] (0.0679)
	100	0.8987	0.0222	[0.8716, 0.9238] (0.0521)	0.8984	0.0222	[0.8708, 0.9225] (0.0516)
(2, 4)	10	0.8679	0.0743	[0.7566, 0.9508] (0.1942)	0.8557	0.0700	[0.7480, 0.9369] (0.1889)
	30	0.8677	0.0724	[0.8059, 0.9186] (0.1126)	0.8623	0.0697	[0.7958, 0.9162] (0.1204)
	50	0.8676	0.0721	[0.8174, 0.9096] (0.0921)	0.8649	0.0697	[0.8177, 0.9061] (0.0883)
	100	0.8675	0.0718	[0.8327, 0.8963] (0.0635)	0.8670	0.0697	[0.8308, 0.8976] (0.0667)

TABLE V
The results for the estimates for the different methods of estimation, confidence and credible intervals for the real data

MLE	LSE	CVME	Bayesian	
			Prior I	Prior II
0.6450 [0.4171, 0.8730]	0.8138 [0.6754, 0.8846]	0.6315 [0.3981, 0.8650]	0.5285 [4.7088×10^{-5}, 0.9979]	0.4983 [0.0732, 0.8846]

XI. Conclusion

The multicomponent stress-strength reliability model with the exponentiated generalized inverse Rayleigh distribution was introduced. The estimators for the multicomponent stress-strength reliability function using different methods of estimation were discussed. The asymptotic, parametric bootstrap sampling and student̓s t bootstrap sampling confidence intervals for the multicomponent stress-strength reliability were introduced. Bayesian estimation method and the credible interval for the multicomponent stress-strength reliability function in case of known and unknown parameters were presented. A simulation study was introduced to show the results for the different methods of estimation for the multicomponent stress-strength reliability. A real data application was introduced to show the results for the multicomponent stress-strength model.

Acknowledgment

The author would like to thank Prince Sattam Bin Abdulaziz University for their valuable help.

References

[1] G. K. Bhattacharyya and R. A. Johnson, "Estimation of reliability in a multicomponent stress-strength model, Journal of the American Statistical Association," vol. 69, no. 348, pp. 966-970, 1974, DOI: 10.1080/01621459.1974.10480238

[2] R. D. Norman and I. Guttman, "Bayesian analysis of reliability in multicomponent stress-strength models," Communications in Statistics-Theory and Methods, vol. 7, no. 5, pp. 441-451, 1978, DOI: 10.1080/03610927808827637

[3] G. S. Rao and R.R.L. Kantam, "Estimation of reliability in multicomponent stress strength model: log-logistic distribution," Electronic Journal of Applied Statistical Analysis, vol. 3, no. 2, pp. 75-84, 2010, DOI: 10.1285/i20705948v3n2p75

[4] G. S. Rao, "Estimation of reliability in multicomponent stress-strength based on generalized exponential distribution," Revista Colombiana de Estadística, vol. 35, no. 1, pp. 67-76, 2012.

[5] G. S. Rao, M. Aslam and O. H. Arif, "Estimation of reliability in multicomponent stress–strength based on two parameter exponentiated Weibull Distribution," Communications in Statistics-Theory and Methods, vol. 46, no. 15, pp. 7495-7502, 2017, DOI: 10.1080/03610926.2016.1154155

[6] S. Dey, J. Mazucheli and M. Z. Anis, "Estimation of reliability of multicomponent stress–strength for a Kumaraswamy distribution," Communications in Statistics - Theory and Methods, vol. 46, no. 4, pp. 1560-1572, 2017, DOI: 10.1080/03610926.2015.1022457

[7] F. Kızılaslan and M. Nadar, "Estimation of reliability in a multicomponent stress–strength model based on a bivariate Kumaraswamy distribution," Statistical Papers, vol. 59, pp. 307–340, 2018, https://doi.org/10.1007/s00362-016-0765-8

[8] M. KH. Hassan and M. I. Alohali, "Estimation of reliability in a multicomponent stress-strength model based on generalized linear failure rate distribution," Hacettepe Journal of Mathematics and Statistics, vol. 47, no. 6, pp. 1634 –1651, 2018.

[9] F. Kızılaslan, "Classical and Bayesian estimation of reliability in a multicomponent stress–strength model based on a general class of inverse exponentiated distributions," Statistical Papers, vol. 59, pp. 1161–1192, 2018, https://doi.org/10.1007/s00362-016-0810-7

[10] A. Kohansal, "On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample," Statistical Papers, vol. 60, pp. 2185–2224, 2019, https://doi.org/10.1007/s00362-017-0916-6

[11] F. G. Akgül, "Reliability estimation in multicomponent stress–strength model for Topp-Leone distribution," Journal of Statistical

Computation and Simulation, vol. 89, no. 15, pp. 2914-2929, 2019, DOI: 10.1080/00949655.2019.1643348

[12] G. S. Rao, F. A. Bhatti, M. Aslam and M. Albassam, "Estimation of reliability in a multicomponent stress–strength system for the exponentiated moment-based exponential distribution," Algorithms, vol. 12, no. 12, pp. 246, 2019, https://doi.org/10.3390/a12120246

[13] M. K. Jha, S. Dey and Y. M. Tripathi, "Reliability estimation in a multicomponent stress–strength based on unit-Gompertz distribution," International Journal of Quality & Reliability Management, vol. 37, no. 3, pp. 428-450, 2020, DOI: 10.1108/IJQRM-04-2019-0136

[14] A. S. Hassan, H. F. Nagy, H. Z. Muhammed and M. S. Saad, "Estimation of multicomponent stress-strength reliability following Weibull distribution based on upper record values," Journal of Taibah University for Science, vol. 14, no. 1, pp. 244-253, 2020, https://doi.org/10.1080/16583655.2020.1721751

[15] R. M. Alotaibi, Y. M. Tripathi, S. Dey and H. R. Rezk, "Bayesian and non-Bayesian reliability estimation of multicomponent stress–strength model for unit Weibull distribution," Journal of Taibah University for Science, vol. 14, no. 1, pp. 1164-1181, 2020, DOI: 10.1080/16583655.2020.1806525

[16] M. S. Kotb and M. Z. Raqab, "Estimation of reliability for multi-component stress–strength model based on modified Weibull distribution," Statistical Papers, vol. 62, pp. 2763–2797, 2020, https://doi.org/10.1007/s00362-020-01213-0

[17] J. Jia, Z. Yan and X. Peng, "Inferences on stress-strength reliability from inverse Weibull distribution based on first-failure progressively unified hybrid censoring schemes," IAENG International Journal of Applied Mathematics, vol. 51, no. 4, pp. 899-907, 2021.

[18] N. Jana and S. Bera, "Interval estimation of multicomponent stress–strength reliability based on inverse Weibull distribution," Mathematics and Computers in Simulation, vol. 191, pp. 95-119, 2022, https://doi.org/10.1016/j.matcom.2021.07.026

[19] H. H. Ahmad, E. M. Almetwally and D. A. Ramadan, "A comparative inference on reliability estimation for a multi-component stress-strength model under power Lomax distribution with applications," AIMS Mathematics, vol. 7, no. 10, pp. 18050–18079, 2022, DOI: 10.3934/math.2022994

[20] Y. Lio, T-R Tsai, L. Wang and I. P. C. Tejada, "Inferences of the multicomponent stress–strength reliability for Burr XII distributions," Mathematics, vol. 10, no. 14, pp. 2478, 2022, https:// doi.org/10.3390/math10142478

[21] H. Pasha-Zanoosi, A. Pourdarvish and A. Asgharzadeh, "Multicomponent stress-strength reliability with exponentiated Teissier distribution," Austrian Journal of Statistics, vol. 51, pp. 35–59, 2022, http://www.ajs.or.at/ doi:10.17713/ajs.v51i4.1327

[22] L. Zhang, A. Xu, L. An and M. Li, "Bayesian inference of system reliability for multicomponent stress-strength model under Marshall-Olkin Weibull distribution," Systems, vol. 10, no. 6, pp. 196, 2022, https://doi.org/10.3390/systems10060196

[23] K. Fatima, S. Naqash and S.P. Ahmad, Exponentiated generalized inverse Rayleigh distribution with applications in medical sciences, Pakistan Journal of Statistics, vol. 34, no. 5, pp. 425-439, 2018.

[24] J. F. Lawless, Statistical Models and Methods for Lifetime Data. 2nd Edition, John Wiley & Sons, New York, 2023.