Investigating the Effectiveness of the Laplace Residual Power Series Approach for Volterra Integro-Differential Equation

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Abstract—**This paper considers the use of the Laplace residual power series method to solve linear Volterra integro-differential equations. It demonstrates how well the approach holds up to both linear and nonlinear fractional starting value situations. Important contributions are comparing the Laplace residual power series approach with other numerical methods and providing evidence of its reliability through recent investigations.**

Index Terms—**Laplace residual power series, Integro differential equations, Volterra integral equations, nonlinear equations.**

I. Introduction

FUNDAMENTAL to mathematical modeling, integral-differential equations lie at the heart of many scientific and engineering domains [1]. These formulas are often convertible into Volterra or Fredholm integral equations and are especially useful in areas like potential theory and mathematical physics [2]. As shown by [3], their complexity derives from the need to model intricate systems with many independent and dependent variables [4], [5], [6].

An important advancement in linear integro-differential equations was presented by [7] through the use of the Homotopy Analysis Method (HAM). The feasibility of employing more intricate techniques, such as the Laplace Residual Power Series (LRPS) method, has been made possible by this approach. The LRPS system exhibits resilience and adaptability, as evidenced by many inquiries. Shafee [8], Al-Ahmad [9] and Alaroud [10] utilized the application of fractional systems and linear and nonlinear fractional initial value problems to study partial differential equations. The effectiveness and reliability of LRPS have been confirmed by these investigations. Momani [11] devised a residual power series technique to address the issue of solving nonlinear Volterra integral equations in the context of initial value problem systems. On the other hand [12], [13] utilized a power series approach.

A wide variety of numerical techniques have also been put forth in parallel, demonstrating the creativity and adaptability of the area. An effective B-spline collocation and cubature formula-based algorithm was presented by Cravero [14]. By presenting a unique difference technique for linear first-order Volterra delay integro-differential equations, Qazza and Hatamleh [15] made significant progress toward first-order convergence. In Salah [16], the Nystrom approach was used to solve nonlinear integro-differential equations of the Volterra type, yielding information about the existence and uniqueness of the solution. Additionally, Çelik [17] solved a class of Volterra integral equation systems using the differential transform method, and both linear and non-linear systems had exact solutions [18], [19].

These advances bring to light the dynamic character of integro-differential equation research. The many methodologies offer opportunities for cross-disciplinary applications and breakthroughs in addition to offering alternative ways to solve these intricate equations. This wide range of approaches highlights the continued development and significance of mathematical modeling in solving practical issues in various scientific and technical fields [20], [21], [22], [23], [24].

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II. Preliminaries

In this section, we provide an overview of key information about the Laplace transform [25], [26], [27], [28], [29], power series, and integral equation, including some properties essential for this article [30], [31], [32].

Definition 2.1: For a continuous function $g(x)$ defined on $[0,\infty)$ or $G(s)$, the Laplace transform is $\mathcal{L}[g(x)]$.It has the following definition:

$$G(s) = \mathcal{L}[g(x)] = \int_0^{\infty} e^{-sx} g(x)\, dx, \quad s > 0. \tag{1}$$

The existence of the Laplace transform of $g(x)$ is determined by whether the integral (1) converges for any value of s; if it does, the Laplace transform exists; otherwise, it does not exist.

The expression for the inverse Laplace transform is formulated as follows:

$$\mathcal{L}^{-1}[G(s)] = g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}\, G(s)\, ds, \qquad c \in \mathbb{R}. \tag{2}$$

Theorem 2.1: If $g(x)$ is a piecewise continuous function within any finite interval $0 \leq x \leq \beta$ and has exponential order δ for $x > \beta$, satisfying:

$$|g(x)| < \xi e^{x\delta},$$

for some $\xi > 0$, then its Laplace transform $G(s)$ exists for all $s > \delta$.

Theorem 2.2: Assume that $g(x)$ and $h(x)$ are two functions of exponential order, with their Laplace transforms $G(s)$, $H(s)$ respectively, μ_1 and μ_2 are two constants. After that, we have the properties that are listed below:

i. $\mathcal{L}\left[\mu_1 g\left(x\right)+\mu_2 h\left(x\right)\right]=\mu_1\mathcal{L}[g\left(x\right)]+=\mu_2\mathcal{L}[h\left(x\right)]$
$=\mu_1 G(s)+\mu_2 H(s)$.
ii. $\mathcal{L}\left[x^n g(x)\right]=\left(-1\right)^n\frac{d^n}{ds^n}G\left(s\right), \quad n=1,2,\ldots.$
iii. $\lim\limits_{\mu\to\infty} sG\left(s\right)=g\left(0\right).$
iv. $\mathcal{L}\left[\ g(x)*h(x)\right]=\ \mathcal{L}\left[g\left(x\right)\right]\ \mathcal{L}\left[h\left(x\right)\right]=G\left(s\right)H(s)$, where
$g\left(x\right)*h\left(x\right)=\int_0^x g\left(x-\tau\right)h\left(x\right)dx$ (Convolution).
v. $\mathcal{L}\left[g^{(n)}(x)\right]=s^n\mathcal{L}\left[g(x)\right]-\sum_{k=0}^{n-1}s^{n-k-1}g^{(k)}\left(0\right).$

Definition 2.2: An infinite series of the form:

$$\sum_{n=0}^{\infty}c_n\left(x-x_0\right)^n=c_0+c_1\left(x-x_0\right)+c_2\left(x-x_0\right)^2+\ldots, \tag{3}$$

is called a power series about $x=x_0$ where x is a variable and $c_n^{,}s$ are constants called the coefficients of the series.

Definition 2.3: A series that has the following representation

$$\sum_{n=-\infty}^{\infty}c_n x^n=\sum_{n=1}^{\infty}\frac{c_{-n}}{x^n}+\sum_{n=0}^{\infty}c_n x^n, \tag{4}$$

is called Laurent series about $x=0$, where the variable is x and the series' coefficients are $c_n^{\prime}s$. The series $\sum_{n=0}^{\infty}c_n x^n$ is referred to as the Laurent series' analytic or regular component, whereas the series $\sum_{n=1}^{\infty}\frac{c_{-n}}{x^n}$ is referred to as its singular or primary part.

Theorem 2.3: If there exists a power series representation (expansion) for function $g(x)$ centered at x_0 as follow, $g(x)=\sum_{n=0}^{\infty}c_n\left(x-x_0\right)^n$ has radius of convergence $R>0$, then $g(x)$ is infinitely differentiable in $|x-x_0|<R$, and in this case, the formula provides the coefficients for the series $c_n=\frac{g^{(n)}(x_0)}{n!}$, then $g\left(x\right)$ must have the following structure:

$$g\left(x\right)=\sum_{n=0}^{\infty}\frac{g^{(n)}\left(x_0\right)}{n!}\left(x-x_0\right)^n. \tag{5}$$

The series in Equation (5) is called the Taylor series of the function $g(x)$ at x_0. For the special case $x_0=0$, the Taylor series becomes called the Maclaurin series.

Theorem 2.4: If $G\left(s\right)=\mathcal{L}\left[g(x)\right]$ has a Laurent series representation about $s=0$ as follows:

$$G\left(s\right)=\frac{c_0}{s}+\sum_{n=1}^{\infty}\frac{c_n}{s^{n+1}}, \quad s>0, \tag{6}$$

then $c_n=g^{(n)}\left(0\right),\ n=0,1,2,\cdots.$

Definition 2.4: An integral equation is an equation where the unknown function $g(x)$ that needs to be determined it appears within an integral. Integral equations are highly valuable mathematical tools in both pure and applied mathematics.

A typical form of an integral equation in $g(x)$ is of the form:

$$g\left(x\right)=f\left(x\right)+\lambda\int_{a(x)}^{b(x)}k\left(x,\tau\right)g\left(\tau\right)d\tau, \tag{7}$$

where $k(x,\tau)$ is called the kernel of the integral Equation (7), and $a(x)$, $b(x)$ are the limits of the integration.

Definition 2.5: Volterra linear integral equations are commonly represented in the following standard form:

$$h(x)g\left(x\right)=f(x)+\lambda\int_a^x k\left(x,\tau\right)g\left(\tau\right)d\tau, \tag{8}$$

in this equation, the unknown function is $g(x)$ it appears linearly under the integral sign. If $h(x)=1$, the equation is simplified to:

$$g\left(x\right)=f(x)+\lambda\int_a^x k\left(x,\tau\right)g\left(\tau\right)d\tau, \tag{9}$$

this equation is referred to as the Volterra integral equation of the second kind. Conversely, if $h\left(x\right)=0$, the equation becomes:

$$f(x)+\lambda\int_a^x k\left(x,\tau\right)g\left(\tau\right)d\tau=0, \tag{10}$$

Remark 2.1: If $k(x,\tau)=k(x-\tau)$, such that in $(x-\tau)$, $e^{x-\tau}$,...,then the Equation (8) is called the Volterra integral equation of convolution type.

III. LRPS method for solving linear Volterra integro-differential equations

This section is divided into two sections: in the first, we walk through the processes of the LRPS approach for solving integro differential equations, and in the second, we answer a few sample problems to demonstrate how easy the method is to use.

A. Steps of LRPS method

To perform the LRPS technique, for solving equation

$$g^{(n)}\left(x\right)=f\left(x\right)+\lambda\int_0^x k\left(x-\tau\right)g\left(\tau\right)d\tau, \tag{11}$$

where $f\left(x\right)$, $g\left(x\right)$ and $k\left(x\right)$ are analytic functions, with the initial conditions:

$$g\left(0\right)=a_1,\ \ g'\left(0\right)=a_2,\ \ g''\left(0\right)=a_3,\ldots,g^{(n-1)}(0)=a_n.$$

To get the solution, we can follow the steps.

Step 1. Starting with the application of the Laplace transform to both sides of Equation (11), we obtain the following result,

$$\mathcal{L}\left[g^{(n)}\left(x\right)\right]=\mathcal{L}\left[f\left(x\right)\right]+\mathcal{L}\left[\lambda\int_0^x k\left(x-\tau\right)g\left(\tau\right)d\tau\right]. \tag{12}$$

Step 2. Depending on Laplace transform and convolution theorem, the Equation (12), can be rewritten as follows:

$$\begin{aligned}s^n G\left(s\right)&=s^{n-1}g\left(0\right)+s^{n-2}g'\left(0\right)+\ldots+g^{(n-1)}\left(0\right)\\&\quad+F\left(s\right)+\lambda\ K\left(s\right)\ G\left(s\right),\end{aligned} \tag{13}$$

where $G\left(s\right)=\mathcal{L}\left[g\left(x\right)\right]$, $F\left(s\right)=\mathcal{L}\left[f\left(x\right)\right]$, $K(s)=\mathcal{L}[k\left(x\right)]$.

Step 3. Multiplying Equation (13) by $\frac{1}{s^n}$, and utilizing the initial conditions, to simplifying Equation (13) into the following form:

$$G\left(s\right)=\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\ \frac{F\left(s\right)}{s^n}+\frac{1}{s^n}\lambda\ K\left(s\right)\ G\left(s\right). \tag{14}$$

Step 4. Examining the solution to Equation (14) which takes on the following structure:

$$G\left(s\right)=\sum_{i=0}^{\infty}\frac{c_i}{s^{i+1}}, \quad s>0. \tag{15}$$

Step 5. Applying the initial condition by credit Theorem 2.4, we can determine the first n- coefficients of the previous structure, so the Equation (15) can be written as follows:

$$G(s)=\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\sum_{i=n}^{\infty}\frac{c_i}{s^{i+1}}\ ,\quad s>0, \tag{16}$$

and the μ-th truncated series of (16) is given by:

$$G_\mu(s)=\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i}{s^{i+1}},\quad s>0, \tag{17}$$

where $\mu=n,n+1,\ldots$.

Step 6. Evaluating the Laplace residual function from Equation (14) and the μth-truncated Laplace residual function independently,

$$\begin{aligned}LRes(s)&=G(s)\\&-\left(\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\frac{F(s)}{s^n}+\frac{1}{s^n}\lambda\ K(s)\,G(s)\right),\end{aligned} \tag{18}$$

$$\begin{aligned}LRes_\mu(s)&=G_\mu(s)\\&-\left(\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\ \frac{F(s)}{s^n}+\frac{1}{s^n}\lambda\ K(s)\,G_\mu(s)\right).\end{aligned} \tag{19}$$

Step 7. Substituting the sum of $G_\mu(s)$ into (17) in place of the term (19) to get:

$$\begin{aligned}LRes_\mu(s)&=\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i}{s^{i+1}}\\&-\left(\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\frac{F(s)}{s^n}+\frac{1}{s^n}\lambda K(s)\right.\\&\left.\left(\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i}{s^{i+1}}\right)\right).\end{aligned} \tag{20}$$

Step 8. Multiplying both sides of Equation (20) by $s^{\mu+1}$ and then take the limit as s approaches infinity, here in we need the following facts that can be found in [30]:

i. $\lim_{\mu\to\infty}LRes_\mu(s)=LRes(s)\,,LRes(s)=0,$ for all $s>0,$

ii. $\lim_{s\to\infty}(s\ LRes(s))=0$, which implies $\lim_{s\to\infty}(sLRes_\mu(s))=0,$

iii. $\lim_{s\to\infty}\left(s^{\mu+1}LRes(s)\right)=\lim_{s\to\infty}\left(s^{\mu+1}LRes_\mu(s)\right)=0,$ $\mu=1,2,3,\cdots,$

to obtain:

$$\begin{aligned}&\underbrace{\lim_{s\to\infty}s^{\mu+1}LRes_\mu(s)}_{=0}=\lim_{s\to\infty}s^{\mu+1}\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}\\&+\lim_{s\to\infty}s^{\mu+1}\sum_{i=n}^{\mu}\frac{c_i}{s^{i+1}}-\lim_{s\to\infty}s^{\mu+1}\left(\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}\right.\\&\left.+\frac{F(s)}{s^n}+\frac{1}{s^n}\lambda K(s)\left(\sum_{i=0}^{n-1}\frac{a_{i+1}}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i}{s^{i+1}}\right)\right).\end{aligned} \tag{21}$$

Step 9. Determining the values of the coefficient c_i's in Equation (21) by solving the system in Equation (21) for $\mu=1,2,3,\cdots$, recursively.

Step 10. In order to determine the approximate solution, the estimated values of c_i's are substituted into the truncated series of $G(s)$.

Step 11. Applying the inverse Laplace transform to $G(s)$ in order to obtain the approximate solution of $g(x)$, which is the solution of Equation (11).

B. Applications

Example 3.1. Consider the integro-differential equation of the form:

$$g''(x)=1+\int_0^x(x-\tau)\,g(\tau)\,d\tau, \tag{22}$$

with the initial conditions $g(0)=-1$, $g'(0)=0$.

Solution. Utilizing the Laplace transform to both aspects of Equation (22)

$$\mathcal{L}\left[g''(x)\right]=\mathcal{L}\left[1\right]+\mathcal{L}\left[\int_0^x(x-\tau)g(\tau)\,d\tau\right]. \tag{23}$$

Running Laplace transform and using convolution theory of Laplace transform to get:

$$s^2G(s)-s\ g(0)-g'(0)=\frac{1}{s}+\frac{1}{s^2}G(s)\,. \tag{24}$$

Multiplying Equation (24) by $\frac{1}{s^2}$, and utilizing the initial conditions to simplify Equation (24) into the following form:

$$G(s)=\frac{1}{s}+\frac{1}{s^3}+\frac{1}{s^4}G(s)\,. \tag{25}$$

Assume that

$$G(s)=\sum_{i=0}^{\infty}\frac{c_i}{s^{i+1}},\quad s>0, \tag{26}$$

applying the initial condition, based on the Theorem 2.4, so the Equation (26) can be written as follows:

$$G(s)=\frac{1}{s}+\sum_{i=2}^{\infty}\frac{c_i}{s^{i+1}}\ ,\quad s>0, \tag{27}$$

and the μth-truncated series of (27) is given by:

$$G_\mu(s)=\frac{1}{s}+\sum_{i=2}^{\mu}\frac{c_i}{s^{i+1}},\quad s>0, \tag{28}$$

we define the Laplace residual function of Equation (25) as follows:

$$LRes\ \ (s)=G(s)-\frac{1}{s}-\frac{1}{s^3}-\frac{1}{s^4}\ G(s)\,, \tag{29}$$

and

$$LRes_\mu(s)=G_\mu(s)-\frac{1}{s}-\frac{1}{s^3}-\frac{1}{s^4}\ G_\mu(s)\,, \tag{30}$$

to find the second coefficient c_2, we define the second truncated series $G_2(s)$ as: $G_2(s)=\frac{1}{s}+\frac{c_2}{s^3}$, substituting $G_2(s)$ in the second Laplace residual function $LRes_2(s)$, to get:

$$LRes_2(s)=\frac{1}{s}+\frac{c_2}{s^3}-\frac{1}{s}-\frac{1}{s^3}-\frac{1}{s^4}\left(\frac{1}{s}+\frac{c_2}{s^3}\right). \tag{31}$$

Multiplying both sides by s^3, then we get:

$$s^3LRes_2(s)=c_2-1-\frac{s^3}{s^4}\left(\frac{1}{s}+\frac{c_2}{s^3}\right). \tag{32}$$

By taking the limit to both sides as $s \to \infty$, we get the value of $c_2 : c_2 = 1$.

Thus, the second approximation of the solution of Equation (25) is:

$$G_2(s) = \frac{1}{s} + \frac{1}{s^3}.$$

Following the same steps to calculate c_3, we define the third truncated series $G_3(s)$ as:

$$G_3(s) = \frac{1}{s} + \frac{1}{s^3} + \frac{c_3}{s^4},$$

substituting $G_3(s)$ in the third Laplace residual function $LRes_3(s)$, to get:

$$LRes_3(s) = \frac{1}{s} + \frac{1}{s^3} + \frac{c_3}{s^4} - \frac{1}{s} - \frac{1}{s^3} - \frac{1}{s^4}\left(\frac{1}{s} + \frac{1}{s^3} + \frac{c_3}{s^4}\right). \tag{33}$$

Multiplying both sides by s^4, and taking the limit to both sides as $s \to \infty$, we get the value of c_3: $c_3 = 0$, then the third approximation of Equation (25) is:

$$G_3(s) = \frac{1}{s} + \frac{1}{s^3}.$$

The values of the coefficients can be determined by using the same procedures:

$$c_4 = c_6 = c_{2n} \ldots = 1, \text{where} \quad n = 1, 2, 3, \ldots,$$

$$c_5 = c_7 = c_{2n+1} \ldots = 0, \text{where} \quad n = 1, 2, 3, \cdots.$$

To find the solution $g(x)$, of Equation (22), we operate the inverse Laplace transform to $G(s)$, to get:

$$\begin{aligned} g(x) = \mathcal{L}^{-1}[G(s)] &= \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s^3} + \frac{1}{s^5} + \frac{1}{s^7} + \ldots\right] \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{2n!} + \ldots = \cosh x. \end{aligned}$$

Example 3.2. Consider the integro-differential equation of the form:

$$g\prime\prime(x) = x + \int_0^x (x - \tau)\, g(\tau)\, d\tau, \tag{34}$$

with the initial conditions

$$g(0) = 0, \qquad g\prime(0) = 1 \tag{35}$$

Solution. Utilizing the Laplace transform to both aspects of Equation (34) Running Laplace transform and using convolution theory of Laplace transform to get:

$$\mathcal{L}[g\prime\prime(x)] = \mathcal{L}[x] + \mathcal{L}\left[\int_0^x (x-\tau) g(\tau)\, d\tau\right]. \tag{36}$$

Running Laplace transform and using convolution theory of Laplace transform to get:

$$s^2 G(s) - s\; g(0) - g'(0) = \frac{1}{s^2} + \frac{1}{s^2} G(s). \tag{37}$$

Multiplying previous equation by $\frac{1}{s^2}$, and utilizing the initial conditions to simplify Equation (37) into the following form:

$$G(s) = \frac{1}{s^2} + \frac{1}{s^4} + \frac{1}{s^4} G(s). \tag{38}$$

Assume that:

$$G(s) = \sum_{i=0}^{\infty} \frac{c_i}{s^{i+1}}, \quad s > 0. \tag{39}$$

Applying the initial condition, based on the Theorem 2.4 so the Equation (39) can be written as follows:

$$G(s) = \frac{1}{s^2} + \sum_{i=2}^{\infty} \frac{c_i}{s^{i+1}}, \; s > 0, \tag{40}$$

and the μ th-truncated series of (40) is given by:

$$G_\mu(s) = \frac{1}{s^2} + \sum_{i=2}^{\mu} \frac{c_i}{s^{i+1}}, \quad s > 0. \tag{41}$$

We define the Laplace residual function of Equation (38) as follows:

$$LRes(s) = G(s) - \frac{1}{s^2} - \frac{1}{s^4} - \frac{1}{s^4} G(s), \tag{42}$$

and

$$LRes_\mu(s) = G_\mu(s) - \frac{1}{s^2} - \frac{1}{s^4} - \frac{1}{s^4} G_\mu(s). \tag{43}$$

To find the second coefficient c_2, we define the second truncated series $G_2(s)$ as:

$$G_2(s) = \frac{1}{s^2} + \frac{c_2}{s^3},$$

substituting $G_2(s)$ in the second Laplace residual function $LRes_2(s)$, to get:

$$LRes_2(s) = \frac{1}{s^2} + \frac{c_2}{s^3} - \frac{1}{s^2} - \frac{1}{s^4} - \frac{1}{s^4}\left(\frac{1}{s^2} + \frac{c_2}{s^3}\right). \tag{44}$$

Multiplying both sides by s^3, then we get:

$$s^3 LRes_2(s) = c_2 - \frac{s^3}{s^4} - \frac{s^3}{s^4}\left(\frac{1}{s^2} + \frac{c_2}{s^3}\right). \tag{45}$$

By taking the limit to both sides as $s \to \infty$, we get the value of $c_2 : c_2 = 0$. Thus, the second approximation of the solution of Equation (38) is:

$$G_2(s) = \frac{1}{s^2}.$$

For series (41), we can identify more coefficients by following the previously described technique. These include $c_3 = 1$, $c_4 = c_6 = c_{2n} \ldots = 0$, and others. $\ldots = 1$ for $c_5 = c_7 = c_{2n+1}$. Thus, the following is the form of the series solution to Equation (38):

$$G(s) = \frac{1}{s^2} + \frac{1}{s^4} + \frac{1}{s^6} + \frac{1}{s^8} + \cdots. \tag{46}$$

There for, the solution of Equation (34) can be expressed in the following series form:

$$g(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

which is the expansion of the exact solution $g(x) = \sinh x$.

IV. LRPS method for solving nonlinear Volterra integro-differential equations

This section consists of two parts, the first one includes the steps of the LRPS method for solving nonlinear Volterra integro differential equations, and in the other section we solve some illustrative examples to show the simplicity of the method.

A. Steps of LRPS method

To perform the LRPS technique, for solving equation

$$g^{(n)}(x) = f(x) + \lambda \int_0^x k(x-\tau)\ h(g(\tau))\, d\tau, \tag{47}$$

where $f(x)$, $g(x)$ and $k(x)$ are analytic functions, with the initial conditions:

$$g(0) = a_1,\ \ g'(0) = a_2,\ \ g''(0) = a_3, \ldots, g^{(n-1)}(0) = a_n. \tag{48}$$

To solve this equation, applying the LRPS technique we follow the steps.

Step 1. Starting with the application of the Laplace transform to both sides of Equation (48), we obtain the following result,

$$\mathcal{L}\left[g^{(n)}(x)\right] = \mathcal{L}[f(x)] + \mathcal{L}\left[\lambda \int_0^x k(x-\tau)\, h(g(\tau))\, d\tau\right]. \tag{49}$$

Step 2. Depending on Laplace transform and convolution theorem, the Equation (49), can be rewritten as follows:

$$\begin{aligned} s^n G(s) &= s^{n-1} g(0) + s^{n-2} g'(0) + \ldots + g^{(n-1)}(0) \\ &\quad + F(s) + \lambda\ K(s)\ \mathcal{L}\left[h\left(\left[\mathcal{L}^{-1}[G(s)]\right]\right)\right], \end{aligned} \tag{50}$$

where $G(s) = \mathcal{L}[g(x)]$, $F(s) = \mathcal{L}[f(x)]$, $K(s) = \mathcal{L}[k(x)]$.

Step 3. Multiplying Equation (50) by $\frac{1}{s^n}$, and utilizing the initial conditions, to simplifying Equation (50) into the following form:

$$\begin{aligned} G(s) &= \sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \frac{F(s)}{s^n} \\ &\quad + \frac{1}{s^n} \lambda\ K(s)\ \mathcal{L}\left[h\left(\left[\mathcal{L}^{-1}[G(s)]\right]\right)\right]. \end{aligned} \tag{51}$$

Step 4. Examining the solution to Equation (51) which takes on the following structure:

$$G(s) = \sum_{i=0}^{\infty} \frac{c_i}{s^{i+1}}, \quad s > 0. \tag{52}$$

Step 5. Applying the initial condition by credit Theorem 2.4, we can determine the first n- coefficients of the previous structure, so the Equation (52) can be written as follows:

$$G(s) = \sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \sum_{i=n}^{\infty} \frac{c_i}{s^{i+1}}\ , \quad s > 0, \tag{53}$$

and the μ-th truncated series of (53) is given by:

$$G_\mu(s) = \sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \sum_{i=n}^{\mu} \frac{c_i}{s^{i+1}}, \quad s > 0, \tag{54}$$

where $\mu = n, n+1, \ldots$.

Step 6. Evaluating the Laplace residual function from Equation (51) and the μth-truncated Laplace residual function independently,

$$\begin{aligned} LRes(s) &= G(s) \\ &\quad - \left(\sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \frac{F(s)}{s^n} \right. \\ &\quad \left. + \frac{1}{s^n} \lambda\ K(s)\, \mathcal{L}\left[h\left(\mathcal{L}^{-1}[G(s)]\right)\right] \right), \end{aligned} \tag{55}$$

$$\begin{aligned} LRes_\mu(s) &= G_\mu(s) \\ &\quad - \left(\sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \frac{F(s)}{s^n} \right. \\ &\quad \left. + \frac{1}{s^n} \lambda\ K(s)\, \mathcal{L}\left[h\left(\mathcal{L}^{-1}[G_\mu(s)]\right)\right] \right). \end{aligned} \tag{56}$$

Step 7. Substituting the sum of $G_\mu(s)$ into (54) in place of the term (56) to get:

$$\begin{aligned} LRes_\mu(s) &= \sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \sum_{i=n}^{\mu} \frac{c_i}{s^{i+1}} \\ &\quad - \left(\sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \frac{F(s)}{s^n} + \frac{1}{s^n} \lambda K(s) \right. \\ &\quad \left. \mathcal{L}\left[h\left(\mathcal{L}^{-1}\left[\sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \sum_{i=n}^{\mu} \frac{c_i}{s^{i+1}}\right]\right)\right]\right). \end{aligned} \tag{57}$$

Step 8. Multiplying both sides of Equation (57) by $s^{\mu+1}$ and then take the limit as s approaches infinity, here in we need the following facts that can be found in [30]:

i. $\lim_{\mu\to\infty} LRes_\mu(s) = LRes(s), LRes(s) = 0,$ for all $s > 0$,

ii. $\lim_{s\to\infty} (s\ LRes(s)) = 0$, which implies $\lim_{s\to\infty} (sLRes_\mu(s)) = 0$,

iii. $\lim_{s\to\infty} \left(s^{\mu+1} LRes(s)\right) = \lim_{s\to\infty} \left(s^{\mu+1} LRes_\mu(s)\right) = 0,$ $\mu = 1, 2, 3, \cdots,$

to obtain:

$$\begin{aligned} \underbrace{\lim_{s\to\infty} s^{\mu+1} LRes_\mu(s)}_{=0} &= \lim_{s\to\infty} s^{\mu+1} \sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} \\ &\quad + \lim_{s\to\infty} s^{\mu+1} \sum_{i=n}^{\mu} \frac{c_i}{s^{i+1}} - \lim_{s\to\infty} s^{\mu+1} \left(\sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} \right. \\ &\quad + \frac{F(s)}{s^n} \\ &\quad \left. + \frac{1}{s^n} \lambda K(s)\, \mathcal{L}\left[h\left(\mathcal{L}^{-1}\left[\sum_{i=0}^{n-1} \frac{a_{i+1}}{s^{i+1}} + \sum_{i=n}^{\mu} \frac{c_i}{s^{i+1}}\right]\right)\right]\right). \end{aligned} \tag{58}$$

Step 9. Determining the values of the coefficient c_i's in Equation (58) by solving the system in Equation (58) for $\mu = 1, 2, 3, \cdots$, recursively.

Step 10. In order to determine the approximate solution, the estimated values of c_i's are substituted into the truncated series of $G(s)$.

Step 11. Applying the inverse Laplace transform to $G(s)$ in order to obtain the approximate solution of $g(x)$, which is the solution of Equation (47).

B. Applications

Example 3.3. Examine the nonlinear Volterra integro-differential equation in the given form:

$$g'(x) = \frac{3}{2} e^x - \frac{1}{2} e^{3x} + \int_0^x e^{(x-\tau)} g^3(\tau)\, d\tau, \tag{59}$$

with the initial conditions $g(0) = 1$.

Solution. Utilizing the Laplace transform to both aspects of

Equation (59)

$$\mathcal{L}\left[g'(x)\right]=\mathcal{L}\left[\frac{3}{2}e^{x}\right]-\mathcal{L}\left[\frac{1}{2}e^{3x}\right]+\mathcal{L}\left[\int_{0}^{x}e^{(x-\tau)}g^{3}(\tau)\,d\tau\right]. \quad (60)$$

Running Laplace transform and using convolution theory of Laplace transform to get:

$$\begin{aligned} sG(s)-g(0)&=\frac{3}{2(s-1)}-\frac{1}{2(s-3)}\\ &+\frac{1}{s-1}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G(s)\right]\right)^{3}\right]. \end{aligned} \quad (61)$$

Multiplying previous Equation by $\frac{1}{s}$, and utilizing the initial conditions to simplifying Equation (61) into the following form:

$$\begin{aligned} G(s)&=\frac{1}{s}+\frac{3}{2s(s-1)}-\frac{1}{2s(s-3)}\\ &+\frac{1}{s(s-1)}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G(s)\right]\right)^{3}\right]. \end{aligned} \quad (62)$$

Assume that:

$$G(s)=\sum_{i=0}^{\infty}\frac{c_i}{s^{i+1}},\quad s>0. \quad (63)$$

Applying the initial condition as stated in Theorem 2.4, so the Equation (63) can be written as follows:

$$G(s)=\frac{1}{s}+\sum_{i=1}^{\infty}\frac{c_i}{s^{i+1}},\ s>0, \quad (64)$$

and the μth-truncated series of (64) is given by:

$$G_\mu(s)=\frac{1}{s}+\sum_{i=1}^{\mu}\frac{c_i}{s^{i+1}},s>0, \quad (65)$$

We define the Laplace residual function of Equation (62) as follows:

$$\begin{aligned} LRes(s)&=G(s)-\frac{1}{s}-\frac{3}{2s(s-1)}+\frac{1}{2s(s-3)}\\ &-\frac{1}{s(s-1)}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G(s)\right]\right)^{3}\right], \end{aligned} \quad (66)$$

and

$$\begin{aligned} LRes_\mu(s)&=G_\mu(s)-\frac{1}{s}-\frac{3}{2s(s-1)}+\frac{1}{2s(s-3)}\\ &-\frac{1}{s(s-1)}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G_\mu(s)\right]\right)^{3}\right]. \end{aligned} \quad (67)$$

To find the first coefficient c_1, we define the first truncated series $G_1(s)$ as: $G_1(s)=\frac{1}{s}+\frac{c_1}{s^2}$, substituting $G_1(s)$ in the first Laplace residual function $LRes_1(s)$, to get:

$$\begin{aligned} LRes_1(s)&=\frac{1}{s}+\frac{c_1}{s^2}-\frac{1}{s}-\frac{3}{2s(s-1)}\\ &+\frac{1}{2s(s-3)}\\ &-\frac{1}{s(s-1)}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{c_1}{s^2}\right]\right)^{3}\right]. \end{aligned} \quad (68)$$

Simplifying the right side of previous equation, then multiplying both sides by s^2 we obtain:

$$\begin{aligned} s^2LRes_1(s)&=\frac{3s^2}{s^2(s-1)(s-3)}+c_1\\ &-\frac{s^2}{s(s-1)}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{c_1}{s^2}\right]\right)^{3}\right]. \end{aligned} \quad (69)$$

By taking the limit to both sides as $s\to\infty$, we get the value of c_1, $c_1=1$. Thus, the first approximation of the solution of Equation (62) is:

$$G_1(s)=\frac{1}{s}+\frac{1}{s^2}.$$

Following the same steps to calculate c_2, we define the second truncated series $G_2(s)$ as: $G_2(s)=\frac{1}{s}+\frac{1}{s^2}+\frac{c_2}{s^3}$, substituting $G_2(s)$ in the second Laplace residual function $LRes_2(s)$, to get:

$$\begin{aligned} LRes_2(s)&=\frac{1}{s}+\frac{1}{s^2}+\frac{c_2}{s^3}\\ &-\frac{1}{s}-\frac{3}{2s(s-1)}+\frac{1}{2s(s-3)}\\ &-\frac{1}{s(s-1)}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{1}{s^2}+\frac{c_2}{s^3}\right]\right)^{3}\right]. \end{aligned} \quad (70)$$

Simplifying the right side of previous equation, then multiplying both sides by s^3, then taking the limit to both sides as $s\to\infty$, we get the value of $c_2=1$. Thus, the solution of Equation (62), in its second approximation, is:

$$G_2(s)=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}.$$

Repeating the same steps, the values of the coefficient can be found in the manner described below.:

$$c_3=c_4=\ldots=1.$$

To find the solution $g(x)$, of Equation (59), we operate the inverse Laplace transform to $G(s)$, to get:

$$\begin{aligned} g(x)=\mathcal{L}^{-1}\left[G(s)\right]&=\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}+\ldots\right]\\ &=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots=e^x. \end{aligned}$$

Example 3.4. Examine the integro-differential equation in the given form:

$$g'(x)=-1+\int_0^x g^2(\tau)\,d\tau, \quad (71)$$

with the initial conditions $g(0)=0$.

Solution. Utilizing the Laplace transform to both aspects of Equation (71)

$$\mathcal{L}\left[g'(x)\right]=\mathcal{L}\left[-1\right]+\mathcal{L}\left[\int_0^x g^2(\tau)\,d\tau\right]. \quad (72)$$

$$sG(s)-g(0)=\frac{-1}{s}+\frac{1}{s}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G(s)\right]\right)^{2}\right]. \quad (73)$$

Running Laplace transform and using convolution theory of Laplace transform to get: Multiplying previous equation by $\frac{1}{s}$, and utilizing the initial conditions to simplify Equation (73) into the following form:

$$G(s)=\frac{-1}{s^2}+\frac{1}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G(s)\right]\right)^{2}\right]. \quad (74)$$

Assume that:

$$G(s)=\sum_{i=0}^{\infty}\frac{c_i}{s^{i+1}},\quad s>0. \quad (75)$$

Applying the initial condition, as stated in Theorem 2.4, so the Equation (75) can be written as follows:

$$G\left(s\right)=\sum_{i=1}^{\infty}\frac{c_i}{s^{i+1}},\quad s>0,\tag{76}$$

and the μth-truncated series of (76) is given by:

$$G_\mu\left(s\right)=\sum_{i=1}^{\mu}\frac{c_i}{s^{i+1}},\quad s>0.\tag{77}$$

We define the Laplace residual function of Equation (74) as follows:

$$LRes\left(s\right)=G\left(s\right)+\frac{1}{s^2}-\frac{1}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G\left(s\right)\right]\right)^2\right],\tag{78}$$

and

$$LRes_\mu\left(s\right)=G_\mu\ \left(s\right)+\frac{1}{s^2}-\frac{1}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G_\mu\left(s\right)\right]\right)^2\right].\tag{79}$$

To find the first coefficient c_1, we define the first truncated series $G_1\left(s\right)$ as:

$$G_1\left(s\right)=\frac{c_1}{s^2},$$

substituting $G_1\left(s\right)$ in the first Laplace residual function $LRes_1\left(s\right)$, to get:

$$LRes_1\left(s\right)=\frac{c_1}{s^2}+\frac{1}{s^2}-\frac{1}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{c_1}{s^2}\right]\right)^2\right].\tag{80}$$

Simplifying the right side of previous equation, then multiplying both sides by s^2, we obtain:

$$s^2LRes_1\left(s\right)=c_1+1-\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{c_1}{s^2}\right]\right)^2\right].\tag{81}$$

By taking the limit to both sides as $s\to\infty$, we get the value of c_1: $c_1=-1$. Thus, the first approximation of the solution of Equation (74) is:

$$G_1\left(s\right)=-\frac{1}{s^2}.$$

It can easily be calculated $c_2=c_3=c_5=c_6=0$. Following the same steps to calculate c_4 , $c_4=2$. Thus, the fourth approximation of the solution of Equation (74) is:

$$G_4\left(s\right)=-\frac{1}{s^2}+\frac{2}{s^5}.\tag{82}$$

Repeating the same steps, we can find the value of $c_7=-20$. To find the solution $g(x)$, of Equation (71), we operate the inverse Laplace transform to $G(s)$, to get:

$$\begin{aligned}g\left(x\right)&=\mathcal{L}^{-1}\left[G\left(s\right)\right]=\mathcal{L}^{-1}\left[-\frac{1}{s^2}+\frac{2}{s^5}-\frac{20}{s^8}\ +\dots\right]\\&=-x+\frac{x^4}{12}-\frac{x^7}{252}+\dots.\end{aligned}\tag{83}$$

Example 3.5. Examine the integro-differential equation in the given form:

$$\begin{aligned}g'\left(x\right)=&-2\sin x-\frac{1}{3}\cos x-\frac{2}{3}\cos 2x\\&+\int_0^x\cos\left(x-\tau\right)g^2\left(\tau\right)d\tau,\end{aligned}\tag{84}$$

with the initial conditions $g\left(0\right)=1$.

Solution. Utilizing the Laplace transform to both aspects of Equation (84)

$$\mathcal{L}\left[g'\left(x\right)\right]=\mathcal{L}[-2\sin x]+\mathcal{L}\left[\int_0^x\cos\left(x-\tau\right)g^2\left(\tau\right)d\tau\right].\tag{85}$$

Running Laplace transform and using convolution theory of Laplace transform to get:

$$\begin{aligned}sG\left(s\right)-g\left(0\right)=&\frac{-2}{s^2+1}-\frac{s}{3(s^2+1)}-\frac{2s}{3\left(s^2+4\right)}\\&+\frac{s}{s^2+1}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G\left(s\right)\right]\right)^2\right].\end{aligned}\tag{86}$$

Multiplying previous Equation by $\frac{1}{s}$, and utilizing the initial conditions to simplify Equation (86) into the following form:

$$\begin{aligned}G\left(s\right)=&\frac{1}{s}-\frac{2}{s(s^2+1)}-\frac{1}{3(s^2+1)}-\frac{2}{3\left(s^2+4\right)}\\&+\frac{1}{s^2+1}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G\left(s\right)\right]\right)^2\right].\end{aligned}\tag{87}$$

Assume that:

$$G\left(s\right)=\sum_{i=0}^{\infty}\frac{c_i}{s^{i+1}},\quad s>0.\tag{88}$$

Applying the initial condition, as stated in Theorem 2.4 so the Equation (88) can be written as follows:

$$G\left(s\right)=\frac{1}{s}+\sum_{i=1}^{\infty}\frac{c_i}{s^{i+1}},\quad s>0,\tag{89}$$

and the μth-truncated series of (89) is given by:

$$G_\mu\left(s\right)=\frac{1}{s}+\sum_{i=1}^{\mu}\frac{c_i}{s^{i+1}},\quad s\ >\ 0.\tag{90}$$

We define the Laplace residual function of Equation (87) as follows:

$$\begin{aligned}LRes\left(s\right)=&G\left(s\right)-\frac{1}{s}+\frac{2}{s(s^2+1)}+\frac{1}{3(s^2+1)}\\&+\frac{2}{3\left(s^2+4\right)}-\frac{1}{s^2+1}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G\left(s\right)\right]\right)^2\right],\end{aligned}\tag{91}$$

$$\begin{aligned}LRes_\mu\left(s\right)=&G_\mu\left(s\right)-\frac{1}{s}+\frac{2}{s(s^2+1)}+\frac{1}{3(s^2+1)}\\&+\frac{2}{3\left(s^2+4\right)}-\frac{1}{s^2+1}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[G_\mu\left(s\right)\right]\right)^2\right].\end{aligned}\tag{92}$$

The first truncated series $G_1\left(s\right)$ is defined as follows in order to determine the first coefficient c_1:

$$G_1\left(s\right)=\frac{1}{s}+\frac{c_1}{s^2},$$

substituting $G_1\left(s\right)$ in the first Laplace residual function $LRes_1(s)$, to get:

$$\begin{aligned}LRes_1\left(s\right)=&\frac{1}{s}+\frac{c_1}{s^2}-\frac{1}{s}+\frac{2}{s(s^2+1)}+\frac{1}{3(s^2+1)}\\&+\frac{2}{3\left(s^2+4\right)}-\frac{1}{s^2+1}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{c_1}{s^2}\right]\right)^2\right].\end{aligned}\tag{93}$$

By taking the limit to both sides as $s \to \infty$, we get the value of c_1: $c_1 = -1$. Thus, the first approximation of the solution of Equation (87) is:

$$G_1(s) = \frac{1}{s} - \frac{1}{s^2}.$$

By using the previously described method, we can identify additional coefficients for series. (90). Some of them are $c_2 = -1$, $c_3 = 1$, $c_4 = 1$, $c_5 = -1, \ldots$.
To find the solution $g(x)$, of Equation (84), we operate the inverse Laplace transform to $G(s)$, to get:

$$\begin{aligned} g(x) &= \mathcal{L}^{-1}\left[G(s)\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s^4} \ + ...\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^5} - \ldots\right] \\ &+ \mathcal{L}^{-1}\left[-\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{s^6} + \ldots\right] = \\ &\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^5} - \ldots\right] \\ &- \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \ldots\right] \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) \\ &= \cos x - \sin x. \end{aligned} \tag{94}$$

V. Conclusion

this article comprehensively explores the efficacy of the LRPS method in solving Volterra integro-differential equations. Through a critical examination of various studies and methods, it highlights the versatility and reliability of LRPS in tackling both linear and nonlinear problems. The article underscores the importance of innovative numerical methods in mathematical modeling, demonstrating the continuous evolution in this field [33], [34], [35]. These insights not only contribute to the academic discourse but also offer practical solutions for complex equations in scientific and engineering applications [36], [37], [38], [39].

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