

# Dynamic Behaviors of a Two Species Amensalism Model with a Second Species Dependent Cover

Yanbo Chong, Qun Zhu, Qianqian Li, and Fengde Chen

**Abstract**—We propose a class of amensalism population models in which the refuge is related to the density of the second population. For the autonomous case, we obtain precise thresholds that guarantee the extinction or stable survival of the first population. For the non-autonomous case, sufficient conditions are obtained to ensure the system's persistence, global asymptotical stability, and extinction, respectively. We demonstrate the feasibility of the main results with the help of numerical simulations.

**Index Terms**—Refuge; Ammensalism; Local stability; Global stability.

## I. INTRODUCTION

**A**MENSALISM is a phenomenon that describes the relationship between two species, wherein one species has limitations and constraints while the other species remains unaffected. For example, grasshoppers and grassland caterpillars[1], the Spanish ibex and the weevil[2] form the amensalism relationship. Over the last twenty years, there has been a significant focus on investigating the dynamic behaviors of the amensalism system; see [1]-[30] and the references cited therein. Such topics as the discrete amensalism model ([6], [9], [13], [21], [23], [30]), the influence of the Allee effect ([5], [8], [12], [14], [15], [16], [17], [22]), the influence of harvesting ([7], [11], [24]), the influence of delay ([10], [25]), the influence of stage structure ([18]), the influence of refuge ([19], [20], [23], [26]), the influence of functional response ([27], [28], [29]), the existence of positive periodic solution [31] etc. were extensively studied.

In 2003, Sun[3] proposed a two-species amensalism model. Zhu and Chen[4] investigated the following amensalism model:

$$\begin{aligned} \frac{dx}{dt} &= x(a_1 - b_1x - c_1y), \\ \frac{dy}{dt} &= y(a_2 - c_2y). \end{aligned} \quad (1)$$

Using vector field analysis, Zhu and Chen examined the system's trajectory and the stability of each equilibrium point (1).

In numerous instances, the use of refuge has safeguarded a consistent proportion of prey against predation. Over the last twenty years, numerous experts have conducted research

on the impact of sanctuary[32]-[48]. However, all of those studies were focused on the predator-prey system. Only recently did scholars [19], [20], [23], [26] pay attention to the influence of refuge on the amensalism model. The Lotka-Volterra amensalism model with refuges was initially studied by Xie et al.[20] for its equilibrium point's existence and global stability. Wu et al.[26] then looked into the amensalism model's dynamic behaviour with refuges and functional responses. Finally, Liu et al.[19] looked into the effects of nonlinear capture and refuges on the amensalism population model, and their research revealed that, under the right circumstances, there are two saddle-node bifurcations and two transcritical bifurcations. Discrete population models are more realistic when there are comparatively few populations, as recently noted by Zhou et al.[23]. They suggested a discrete amensalism model that included a cover for the initial species, and they demonstrated the possibility of flip bifurcation in the system.

In their study, Xie, Chen, and He [20] examined a two-species amensalism model that incorporates a partial cover mechanism for the first species, enabling it to defend itself against the second species. The mathematical representation of the model is as follows:

$$\begin{aligned} \frac{dx}{dt} &= a_1x(t) - b_1x^2(t) - c_1(1-k)x(t)y(t), \\ \frac{dy}{dt} &= a_2y(t) - b_2y^2(t). \end{aligned} \quad (2)$$

The variables  $a_i$ ,  $b_i$ , and  $c_1$  are positive constants, where  $i$  takes the values 1 and 2. Additionally,  $k$  ( $0 < k < 1$ ) represents a cover for the species  $x$ . The study's authors focused on examining the stability of the system. They demonstrated that when the condition  $0 \leq k < 1 - \frac{a_1b_2}{a_2c_1}$  is satisfied, the equilibrium point  $E_2(0, \frac{a_2}{b_2})$  exhibits global stability. Conversely, when the condition  $1 > k > 1 - \frac{a_1b_2}{a_2c_1}$  holds, the equilibrium point  $E_3(x^*, y^*)$  is the only positive equilibrium that demonstrates global stability. To be more specific, it can be said that the local stability criteria of  $E_2(0, \frac{a_2}{b_2})$  are sufficient to guarantee its global stability. Additionally, it can be seen that once a positive equilibrium exists, it is globally stable.

In system (2), it is assumed that a constant proportion of the first species is stated in the cover, and the second species has no influence on this part of the first species. However, generally speaking, the refuge's purpose is to avoid the competition or predation of other species. When the other population does not exist, hiding in the refuge is unnecessary. On the other hand, despite the increase in the number of other species, the number of individuals that the refuge can accommodate is certain. This inspires us to propose refuges

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related to the second species:

$$\frac{dx}{dt} = a_1x(t) - b_1x^2(t) - c_1\left(1 - \frac{ky(t)}{1+y(t)}\right)x(t)y(t), \quad (3)$$

$$\frac{dy}{dt} = a_2y(t) - b_2y^2(t).$$

In this model, we assume that the cover could accumulate  $\frac{ky(t)x(t)}{1+y(t)}$  of the first species. If  $y = 0$ , then the first species need not be stated in the cover; however, as  $y$  approaches infinity, the cover could accumulate at most  $kx$  of the first species. Additionally, the cover shows an increasing function of the second species, suggesting that as the quantity of the second species increases, it encompasses a greater number of first species.

The primary aim of this research is to investigate the stability properties of the various equilibria of the system (2) at both local and global scales while also assessing the influence of refuge. The paper is organized in the following manner: In the subsequent section, we shall examine the presence and local stability characteristics of the equilibria of the system (2). In Section III, we will examine the global stability feature of the system by utilizing differential inequality theory and the Dulac Theorem. In Section IV, we investigate the dynamic behaviors of the non-autonomous system. In Section V, we offer an example and its quantitative simulations to demonstrate the practicality of our key findings.

## II. THE EQUILIBRIA'S EXISTENCE AND STABILITY

The equilibria of the system described in equation (3) are defined by the system

$$\begin{aligned} a_1x - b_1x^2 - c_1\left(1 - k\frac{y}{1+y}\right)xy &= 0, \\ a_2y - b_2y^2 &= 0. \end{aligned} \quad (4)$$

Hence, the system (3) allows for four potential equilibria  $E_1\left(\frac{a_1}{b_1}, 0\right)$ ,  $E_2\left(0, \frac{a_2}{b_2}\right)$ , as well as  $E_3(x^*, y^*)$ , where

$$\begin{aligned} x^* &= \frac{c_1ka_2^2 + a_1a_2b_2 + a_1b_2^2 - c_1a_2^2 - c_1a_2b_2}{b_1b_2(b_2 + a_2)}, \\ y^* &= \frac{a_2}{b_2}. \end{aligned} \quad (5)$$

Obviously,  $E_3$  is a positive equilibrium if and only if

$$\begin{aligned} k &> \frac{c_1a_2^2 + c_1a_2b_2 - a_1a_2b_2 - a_1b_2^2}{a_2^2c_1} \\ &= \left(1 - \frac{a_1b_2}{a_2c_1}\right)\left(1 + \frac{b_2}{a_2}\right) \stackrel{\text{def}}{=} k_0. \end{aligned} \quad (6)$$

Given the four equilibria above, our focus lies on examining their local stability qualities.

**Theorem 2.1.**  $E_0(0, 0)$  and  $E_1\left(\frac{a_1}{b_1}, 0\right)$  are unstable; if  $k < k_0$ , then  $E_2\left(0, \frac{a_2}{b_2}\right)$  is locally asymptotically stable, and if  $k > k_0$ , then  $E_2\left(0, \frac{a_2}{b_2}\right)$  is unstable; if  $k > k_0$  hold,  $E_3(x^*, y^*)$  is locally asymptotically stable.

**Proof.** The calculation of the Jacobian matrix for the system (3) is performed as follows:

$$J(x, y) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & -2b_2y + a_2 \end{pmatrix}, \quad (7)$$

where

$$\begin{aligned} A_{11} &= a_1 - 2b_1x - c_1\left(1 - \frac{ky}{1+y}\right)y, \\ A_{12} &= c_1\left(-\frac{k}{1+y} + \frac{ky}{(1+y)^2}\right)xy - c_1\left(1 - \frac{ky}{1+y}\right)x. \end{aligned} \quad (8)$$

Therefore, the Jacobian matrix of the system (3) evaluated at the equilibrium point  $E_0(0, 0)$  could be expressed as

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Clearly,  $E_0(0, 0)$  is unstable.

For  $E_1\left(\frac{a_1}{b_1}, 0\right)$ , its Jacobian matrix is given by

$$\begin{pmatrix} -a_1 & -\frac{c_1a_1}{b_1} \\ 0 & a_2 \end{pmatrix}.$$

Clearly,  $E_1\left(\frac{a_1}{b_1}, 0\right)$  is unstable.

For  $E_2\left(0, \frac{a_2}{b_2}\right)$ , its Jacobian matrix is given by

$$\begin{pmatrix} \frac{c_1ka_2^2 + a_1a_2b_2 + a_1b_2^2 - c_1a_2^2 - c_1a_2b_2}{b_1b_2(b_2 + a_2)} & 0 \\ 0 & -a_2 \end{pmatrix}.$$

Therefore, if the value of  $k$  is less than  $k_0$ , the system described by equation (3) does not possess a positive equilibrium. In this scenario, the point  $E_2\left(0, \frac{a_2}{b_2}\right)$  is locally asymptotically stable. Conversely, if the value of  $k$  is greater than  $k_0$ , the system does possess a positive equilibrium denoted as  $E_3$ . In this case, the point  $E_2\left(0, \frac{a_2}{b_2}\right)$  is unstable.

Given that  $E_3(x^*, y^*)$  fulfils the equation

$$\begin{aligned} a_1 - b_1x^* - c_1\left(1 - k\frac{y^*}{1+y^*}\right)y^* &= 0, \\ a_2 - b_2y^* &= 0. \end{aligned} \quad (9)$$

The Jacobian matrix concerning the equilibrium point  $E_3$  is provided as follows

$$\begin{pmatrix} -b_1x^* & \Delta_1 \\ 0 & -b_2y^* \end{pmatrix},$$

where

$$\begin{aligned} \Delta_1 &= c_1\left(-\frac{k}{1+y^*} + \frac{ky^*}{(1+y^*)^2}\right)x^*y^* \\ &\quad - c_1\left(1 - \frac{ky^*}{1+y^*}\right)x^*. \end{aligned}$$

If the condition  $k > k_0$  is satisfied, it follows that  $x^* > 0$ . Consequently, both eigenvalues of the matrix mentioned above are negative, and as a result,  $E_3$  exhibits local asymptotic stability. The proof of Theorem 2.1 is concluded at this point.

III. THE GLOBAL STABILITY ANALYSIS OF THE EQUILIBRIA

Subsequently, we will proceed to conduct a more comprehensive analysis of the global stability of the boundary equilibrium  $E_2$ , as well as the positive equilibrium  $E_3$ .

**Theorem 3.1.** *If  $k < k_0$  holds, then  $E_2(0, \frac{a_2}{b_2})$  is globally attractive.*

**Proof.**  $k < k_0$  is equivalent to

$$a_1 < c_1 \left(1 - \frac{k \frac{a_2}{b_2}}{1 + \frac{a_2}{b_2}}\right) \frac{a_2}{b_2} \quad (10)$$

holds. Therefore, one can select a sufficiently small value of  $\varepsilon > 0$  such that the inequality

$$a_1 < c_1 \left(1 - \frac{k(\frac{a_2}{b_2} + \varepsilon)}{1 + (\frac{a_2}{b_2} + \varepsilon)}\right) \left(\frac{a_2}{b_2} - \varepsilon\right) \quad (11)$$

holds. For above  $\varepsilon > 0$ , since the second equation of system (3) is independent of  $x$  and it is a famous Logistic equation, then

$$\lim_{t \rightarrow +\infty} y(t) = \frac{a_2}{b_2}. \quad (12)$$

That is, there exists a positive value  $T > 0$  such that for all  $t > T$ , the following inequality holds

$$\frac{a_2}{b_2} - \varepsilon < y(t) < \frac{a_2}{b_2} + \varepsilon. \quad (13)$$

Now for  $t > T$ , from the first equation of (3), (11), and (13) one has

$$\begin{aligned} \frac{dx}{dt} &= a_1 x(t) - b_1 x^2(t) \\ &\quad - c_1 \left(1 - \frac{ky(t)}{1+y(t)}\right) x(t)y(t), \\ &\leq x(t) \left(a_1 - c_1 \left(1 - \frac{k(\frac{a_2}{b_2} + \varepsilon)}{1 + (\frac{a_2}{b_2} + \varepsilon)}\right) \left(\frac{a_2}{b_2} - \varepsilon\right)\right) \\ &\stackrel{\text{def}}{=} x(t)\Gamma, \end{aligned} \quad (14)$$

where

$$\Gamma = a_1 - c_1 \left(1 - \frac{k(\frac{a_2}{b_2} + \varepsilon)}{1 + (\frac{a_2}{b_2} + \varepsilon)}\right) \left(\frac{a_2}{b_2} - \varepsilon\right) < 0.$$

Therefore, the inequality

$$x(t) < x(T) \exp\{\Gamma(t - T)\} \rightarrow 0 \quad (15)$$

holds as  $t$  approaches positive infinity. Above analysis shows that the boundary equilibrium  $E_2(0, \frac{a_2}{b_2})$  of system (3) is global attractivity. The proof of Theorem 3.1 is concluded at this point.

**Remark 3.1.** Theorem 3.1 shows that if the boundary equilibrium  $E_2$  is locally asymptotically stable, it is also globally attractive.

**Theorem 3.2.** *If  $k > k_0$  hold, then  $E_3(x^*, y^*)$  is globally stable.*

**Proof.** Firstly, similarly to the analysis of (3.6)-(3.8) in [20], It can be demonstrated that all solutions of the system (3)

that originate in  $R_+^2$  exhibit uniform boundedness. In other words, there is an  $\varepsilon > 0$  such that for any  $t$  greater than  $T$

$$x(t) < \frac{a_1}{b_1} + \varepsilon, \quad y(t) < \frac{a_2}{b_2} + \varepsilon. \quad (16)$$

Set  $B = \{(x, y) \mid x < \frac{a_1}{b_1} + \varepsilon, y < \frac{a_2}{b_2} + \varepsilon\}$ . All solutions of the system (3) that start in the positive quadrant are uniformly bounded on the set  $B$ . Note that under the assumption of Theorem 3.2, boundary equilibria  $E_0, E_1$ , and  $E_2$  are all unstable. According to Theorem 2.1, there exists a single local stable positive equilibrium denoted as  $E_3(x^*, y^*)$ . If we could show that the system has no closed orbit in the area  $B$ , then the omega limit set of the system is  $E_3$ , and every solution with positive initial conditions should approach  $E_3$  as  $t \rightarrow +\infty$ . Now, let us consider the Dulac function, denoted as  $u(x, y) = x^{-1}y^{-1}$ , then

$$\begin{aligned} &\frac{\partial(uP)}{\partial x} + \frac{\partial(uQ)}{\partial y} \\ &= \frac{a_1 - 2b_1x - c_1(1 - \frac{ky}{1+y})y}{xy} \\ &\quad - \frac{a_1x - b_1x^2 - c_1(1 - \frac{ky}{1+y})xy}{x^2y} \\ &\quad + \frac{-2b_2y + a_2}{xy} - \frac{-b_2y^2 + a_2y}{x^2y} \\ &= -\frac{b_1x + b_2y}{xy} < 0, \end{aligned} \quad (17)$$

where

$$\begin{aligned} P(x, y) &= a_1x - b_1x^2 - c_1\left(1 - \frac{ky}{1+y}\right)xy, \\ Q(x, y) &= a_2y - b_2y^2. \end{aligned} \quad (18)$$

According to the Dulac Theorem, it may be concluded that a closed orbit within region  $B$  does not exist. The equilibrium  $E_3(x^*, y^*)$  exhibits global asymptotic stability. The proof of Theorem 3.2 is now concluded.

**Remark 3.2.** Theorem 3.2 demonstrates the global stability of the positive equilibrium  $E_3$ , provided that it exists.

**Remark 3.3.** Noting that

$$x^*(k) = \frac{c_1k a_2^2 + a_1a_2b_2 + a_1b_2^2 - c_1a_2^2 - c_1a_2b_2}{b_1b_2(b_2 + a_2)} > 0,$$

then

$$\frac{dx^*(k)}{dk} = \frac{c_1a_2^2}{b_1b_2(b_2 + a_2)} > 0.$$

This implies that the presence of the cover has the potential to enhance the population densities of the first species, thereby reducing the risk of extinction for this species.

**Remark 3.4.** The stability of the second species is demonstrated by (12). For the first species, there is a critical value  $k_0$  such that if the habitat cover is above this threshold ( $k > k_0$ ), the species has the potential to survive. Conversely, the first species will be driven to extinction if the habitat cover is below this threshold ( $k < k_0$ ).

IV. NONAUTONOMOUS CASE

It is well known that the survival environment of a population changes with time. Therefore, a more realistic model needs to take into account the coefficients of the system as a function of time, i.e., it is necessary to consider non-autonomous systems; however, to the best of the author's knowledge, so far, there has not been any scholarly study of non-autonomous population models, in which the refuge is related to the second population, which inspires us to study the non-autonomous scenario of the model (3).

In this section, we will study the non-autonomous case of the system (3)

$$\begin{aligned} \frac{dx}{dt} &= a_1(t)x(t) - b_1(t)x^2(t) \\ &\quad - c_1(t)\left(1 - \frac{ky(t)}{1+y(t)}\right)x(t)y(t), \quad (19) \\ \frac{dy}{dt} &= a_2(t)y(t) - b_2(t)y^2(t). \end{aligned}$$

Throughout this section, for a continuous and bounded function, we let  $f^l = \inf_{t \in R} f(t)$  and  $f^u = \sup_{t \in R} f(t)$ .

In system (19), we always assume:

( $H_1$ )  $k$  is a positive constant which satisfies  $0 < k < 1$ ,  $a_1(t), b_1(t), a_2(t), b_2(t)$  and  $c_1(t)$  are all continuous and strictly positive functions that satisfy

$$\begin{aligned} \min\{a_1^l, b_1^l, a_2^l, b_2^l, c_1^l\} &> 0, \\ \max\{a_1^u, b_1^u, a_2^u, b_2^u, c_1^u\} &< +\infty. \end{aligned}$$

Set

$$y^l \stackrel{def}{=} \frac{a_2^l}{b_2^u}, \quad y^u \stackrel{def}{=} \frac{a_2^u}{b_2^l}. \quad (20)$$

Following, we aim to investigate the persistence and extinction property of the system (19); indeed, we have the following result.

**Theorem 4.1.**

(1) Assuming

$$a_1^u < c_1^l \left(1 - \frac{ky^u}{1+y^u}\right) y^l \quad (21)$$

holds, then the first species  $x(t)$  will be driven to extinction;

(2) Assuming

$$a_1^l > c_1^u \left(1 - \frac{ky^l}{1+y^l}\right) y^u \quad (22)$$

holds, then the system is permanent.

**Proof.** It follows from (21) and (22) that for enough small  $\varepsilon > 0$ , the following inequalities hold:

$$a_1^u < c_1^l \left(1 - \frac{k(y^u + \varepsilon)}{1+(y^u + \varepsilon)}\right) (y^l - \varepsilon), \quad (23)$$

$$a_1^l > c_1^u \left(1 - \frac{k(y^l - \varepsilon)}{1+(y^l - \varepsilon)}\right) (y^u + \varepsilon). \quad (24)$$

From the second equation of system (19) we have

$$y \left(a_2^l - b_2^u y\right) \leq \frac{dy}{dt} \leq y \left(a_2^u - b_2^l y\right), \quad (25)$$

thus, one has

$$y^l \stackrel{def}{=} \frac{a_2^l}{b_2^u} \leq \liminf_{t \rightarrow +\infty} y(t) \leq \limsup_{t \rightarrow +\infty} y(t) \leq \frac{a_2^u}{b_2^l} \stackrel{def}{=} y^u, \quad (26)$$

For  $\varepsilon > 0$  enough small, which satisfies the inequality (23) and (24), there exists a  $T > 0$  such that

$$y^l - \varepsilon < y(t) < y^u + \varepsilon, \quad t \geq T. \quad (27)$$

Now, for  $t \geq T$ , from the first equation of (19) and (27), one has

$$\begin{aligned} \frac{dx}{dt} &= a_1(t)x(t) - b_1(t)x^2(t) \\ &\quad - c_1(t)\left(1 - \frac{ky(t)}{1+y(t)}\right)x(t)y(t) \\ &\leq \left(a_1^u - c_1^l\left(1 - \frac{k(y^u + \varepsilon)}{1+(y^u + \varepsilon)}\right)\right)(y^l - \varepsilon) \\ &\quad - b_1^l x(t)x(t) \end{aligned} \quad (28)$$

If condition (21) holds, then follows from (28) one has

$$x(t) \leq x(T) \exp\left\{\Gamma_1(\varepsilon)(t - T)\right\} \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (29)$$

where

$$\Gamma_1(\varepsilon) = a_1^u - c_1^l \left(1 - \frac{k(y^u + \varepsilon)}{1+(y^u + \varepsilon)}\right) (y^l - \varepsilon) < 0. \quad (30)$$

That is, if (21) holds, the first species  $x(t)$  will be driven to extinction. This ends the proof of Theorem 4.1 (1).

Now assume that inequality (22) holds, then it immediately follows that

$$a_1^u > c_1^l \left(1 - \frac{ky^u}{1+y^u}\right) y^l \quad (31)$$

so, for  $\varepsilon > 0$  enough small, the following inequality holds

$$a_1^u > c_1^l \left(1 - \frac{k(y^u + \varepsilon)}{1+(y^u + \varepsilon)}\right) (y^l - \varepsilon). \quad (32)$$

Hence, it follows from (28) that

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{\Gamma_1(\varepsilon)}{b_1^u}, \quad (33)$$

where  $\Gamma_1(\varepsilon)$  is defined by (30). Setting  $\varepsilon \rightarrow 0$  in (30) leads to

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{\Gamma_1}{b_1^u}, \quad (34)$$

where

$$\Gamma_1 = a_1^u - c_1^l \left(1 - \frac{ky^u}{1+y^u}\right) y^l. \quad (35)$$

Again, from the first equation of (19) and (27), one has

$$\begin{aligned} \frac{dx}{dt} &= a_1(t)x(t) - b_1(t)x^2(t) \\ &\quad - c_1(t)\left(1 - \frac{ky(t)}{1+y(t)}\right)x(t)y(t) \\ &\geq \left(a_1^l - c_1^u\left(1 - \frac{k(y^l - \varepsilon)}{1+(y^l - \varepsilon)}\right)\right)(y^u + \varepsilon) \\ &\quad - b_1^u x(t)x(t) \end{aligned} \quad (36)$$

If condition (22) holds, then follows from (24) one has

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{\Gamma_2(\varepsilon)}{b_1^u}, \quad (37)$$

where

$$\Gamma_2(\varepsilon) = a_1^l - c_1^u \left(1 - \frac{k(y^l - \varepsilon)}{1 + (y^l - \varepsilon)}\right)(y^u + \varepsilon). \quad (38)$$

Setting  $\varepsilon \rightarrow 0$  in (37) leads to

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{\Gamma_2}{b_1^u}, \quad (39)$$

where

$$\Gamma_2 = a_1^l - c_1^u \left(1 - \frac{ky^l}{1 + y^l}\right)y^u. \quad (40)$$

(26), (34), and (39) show that under the assumption (22) holds, the system is permanent. This ends the proof of Theorem 4.1 (2).

The proof of Theorem 4.1 is ended.

Concerned with the global attractivity of the positive solutions of the system, we have the following result.

**Theorem 4.2** *Let  $(x^*(t), y^*(t))$  be a positive solution of system (19). In addition to (22), assume further that the following inequality holds:*

$$b_2^l > \frac{2c_1^u ky^u + c_1^u k(y^u)^2}{(1 + y^l)^2} + c_1^u, \quad (41)$$

where the variables  $y^l, y^u$  are specified by equations (26). Then  $(x^*(t), y^*(t))$  exhibits global asymptotic stability.

**Proof.** The condition expressed in inequality (41) indicates that, for a sufficiently small positive constant  $\varepsilon > 0$  ( $\varepsilon < \frac{1}{2}y^l$ ), the following inequality is valid.

$$b_2^l > \frac{2c_1^u k(y^u + \varepsilon) + c_1^u k(y^u + \varepsilon)^2}{(1 + (y^l - \varepsilon))^2} + c_1^u. \quad (42)$$

Consider the positive solution  $(x(t), u(t))$  of equation (19), it may be deduced from condition (22) and Theorem 4.1 that, given any positive value of  $\varepsilon$ , there exists a positive value of  $T$  such that

$$y^l - \varepsilon < y(t), y^*(t) < y^u + \varepsilon \text{ for all } t \geq T. \quad (43)$$

For  $t \geq T$ , let us consider a Lyapunov function that is defined by

$$V(t) = |\ln\{x(t)\} - \ln\{x^*(t)\}| + |\ln\{y(t)\} - \ln\{y^*(t)\}|. \quad (44)$$

We are now estimating and computing the upper right derivative of  $V(t)$  along the positive solutions of the system

(19) for  $t > T$ . Applying (43) yields the following results:

$$\begin{aligned} & D^+V(t) \\ &= \operatorname{sgn}(x(t) - x^*(t)) \left[ -b_1(t)x(t) + b_1(t)x^*(t) \right. \\ &\quad \left. - c_1(t) \left(1 - \frac{ky(t)}{1 + y(t)}\right)y(t) \right. \\ &\quad \left. + c_1(t) \left(1 - \frac{ky^*(t)}{1 + y^*(t)}\right)y^*(t) \right. \\ &\quad \left. + \operatorname{sgn}(y(t) - y^*(t)) \left[ -b_2(t)y(t) + b_2(t)y^*(t) \right] \right] \\ &= \operatorname{sgn}(x(t) - x^*(t)) \left[ -b_1(t)(x(t) - x^*(t)) \right. \\ &\quad \left. - c_1(t)(y(t) - y^*(t)) \right. \\ &\quad \left. + \frac{c_1(t)k \left[ (y(t))^2(1 + y^*(t)) - (y^*(t))^2(1 + y(t)) \right]}{(1 + y(t))(1 + y^*(t))} \right. \\ &\quad \left. + \operatorname{sgn}(y(t) - y^*(t)) \left[ -b_2(t)(y(t) - y^*(t)) \right] \right] \\ &= \operatorname{sgn}(x(t) - x^*(t)) \left[ -b_1(t)(x(t) - x^*(t)) \right. \\ &\quad \left. - c_1(t)(y(t) - y^*(t)) \right. \\ &\quad \left. + \frac{c_1(t)k(y(t) - y^*(t))(y(t) + y^*(t))}{(1 + y(t))(1 + y^*(t))} \right. \\ &\quad \left. + \frac{c_1(t)ky(t)y^*(t)(y(t) - y^*(t))}{(1 + y(t))(1 + y^*(t))} \right] \\ &\quad \left. + \operatorname{sgn}(y(t) - y^*(t)) \left[ -b_2(t)(y(t) - y^*(t)) \right] \right] \\ &\leq -\Gamma_1|x(t) - x^*(t)| - \Gamma_2|y(t) - y^*(t)|, \end{aligned} \quad (45)$$

where

$$\begin{aligned} \Gamma_1 &= b_1^l > 0, \\ \Gamma_2^\varepsilon &= b_2^l - \frac{2c_1^u k(y^u + \varepsilon) + c_1^u k(y^u + \varepsilon)^2}{(1 + (y^l - \varepsilon))^2} - c_1^u > 0. \end{aligned} \quad (46)$$

For  $t \geq T$ , one thus has

$$D^+V(t) \leq -\mu \left( |x(t) - x^*(t)| + |y(t) - y^*(t)| \right), \quad (47)$$

where  $\mu = \min\{\Gamma_1, \Gamma_2^\varepsilon\}$ . Performing integration on both sides of equation (47) with respect to the variable  $t$  across the interval from  $T$  to  $t$  yields

$$\begin{aligned} V(t) + \mu \int_T^t \left( |x(s) - x^*(s)| + |y(s) - y^*(s)| \right) ds \\ \leq V(T) < +\infty, \quad t \geq T. \end{aligned}$$

Then, for all  $t \geq T$ ,

$$\int_T^t \left( |x(s) - x^*(s)| + |y(s) - y^*(s)| \right) ds \leq \mu^{-1}V(T) < +\infty,$$

and hence,

$$|x(t) - x^*(t)| + |y(t) - y^*(t)| \in L^1([T, +\infty)).$$

The fact that  $x^*(t)$  and  $y^*(t)$  are bounded, and that  $x(t)$  and  $y(t)$  are ultimately bounded, implies that the derivatives of  $x(t)$ ,  $x^*(t)$ ,  $y(t)$ , and  $y^*(t)$  are all bounded for  $t \geq T$ , as

indicated by the equations that govern their behavior. Consequently, it may be inferred that  $|x(t) - x^*(t)| + |y(t) - y^*(t)|$  is uniformly continuous on  $[T, +\infty)$ . Thus,

$$\lim_{t \rightarrow +\infty} (|x(t) - x^*(t)| + |y(t) - y^*(t)|) = 0.$$

The proof is completed.

### V. NUMERIC SIMULATIONS

Now let us consider the following example:

**Example 5.1.** Consider the subsequent system:

$$\begin{aligned} \frac{dx}{dt} &= x(t) - \frac{1}{4}x^2(t) - (1 - k)x(t)y(t), \\ \frac{dy}{dt} &= 2y(t) - y^2(t). \end{aligned} \quad (48)$$

In this system, which corresponds to system (2), we utilize  $a_1 = b_2 = c_1 = 1, a_2 = 2, b_1 = \frac{1}{4}$ . Then from Theorem 3.1 and 3.2 in [20], for  $k < \frac{1}{2}$ , the first species goes extinct while the second species reaches its maximum environmental carrying capacity. For all values of  $k$  larger than  $\frac{1}{2}$ , the system (48) possesses a single positive equilibrium that is globally asymptotically stable.

Now, let us proceed to examine the system

$$\begin{aligned} \frac{dx}{dt} &= x(t) - \frac{1}{4}x^2(t) - \left(1 - k \frac{y}{1+y}\right)x(t)y(t), \\ \frac{dy}{dt} &= 2y(t) - y^2(t). \end{aligned} \quad (49)$$

By simple computation,

$$k_0 = \left(1 - \frac{a_1 b_2}{a_2 c_1}\right) \left(1 + \frac{b_2}{a_2}\right) = \frac{3}{4},$$

thus, from Theorem 3.1, if  $k < \frac{3}{4}$ , the boundary equilibrium  $E_2(0, 2)$  is globally attractive. Fig. 1 shows the feasibility of this assertion (here we choose  $k = \frac{1}{4}$ ), from Theorem 3.2, if  $k > \frac{3}{4}$ , the positive equilibrium  $E_3(x^*, y^*)$  is globally asymptotically stable. Fig. 2 shows the feasibility of this assertion (here, we choose  $k = \frac{7}{8}$ ).

**Example 5.2.** Now, let's consider the following non-autonomous case.

$$\begin{aligned} \frac{dx}{dt} &= \left(\frac{3}{2} - \frac{1}{2} \cos(t)\right)x(t) - \frac{1}{4}x^2(t) \\ &\quad - \left(\frac{3}{16} + \frac{\sin(2t)}{16}\right) \left(1 - \frac{4}{5} \frac{y}{1+y}\right)x(t)y(t), \\ \frac{dy}{dt} &= \left(\frac{3}{2} + \frac{1}{2} \sin(t)\right)y(t) - y^2(t). \end{aligned} \quad (50)$$

Here, corresponding to system (19), we choose  $a_1(t) = \frac{3}{2} - \frac{1}{2} \cos(t)$ ,  $b_1(t) = \frac{1}{4}$ ,  $c_1(t) = \frac{3}{16} + \frac{\sin(2t)}{16}$ ,  $k = \frac{4}{5}$ .  $a_2(t) = \frac{3}{2} + \frac{1}{2} \sin(t)$ ,  $b_2(t) = 1$ . Then, simple computation shows that  $y^l = 1, y^u = 2$ , and

$$a_1^l = 1 > \frac{3}{10} = c_1^u \left(1 - \frac{ky^l}{1+y^l}\right) y^u, \quad (51)$$

$$b_2^l = 1 > \frac{1}{10} + \frac{1}{4} = \frac{2c_1^u ky^u + c_1^u k(y^u)^2}{(1+y^l)^2} + c_1^u. \quad (52)$$

That is, all the conditions of Theorem 4.2 are satisfied, and so, any positive solution  $(x^*(t), y^*(t))$  of system (19) is globally asymptotically stable. Fig. 3 and 4 support this assertion.

### VI. DISCUSSION

This paper proposes an amensalism model in which the refuge is linked to the second species. Our study shows that its dynamics are similar to those of the amensalism model with proportional refuge, and there is a threshold  $k_0$ ; if  $k > k_0$ , then the two populations can coexist stably, and if  $k < k_0$ , then the first population is extinct. However, based on numerical simulations, our proposed model requires a larger refuge space to ensure the two populations' stable coexistence.

It should be pointed out that, to the best of the author's knowledge, this is the first time to investigate the global stability property of a nonautonomous ecosystem with nonlinear refuge. The research methodology of Theorem 4.2 can be used to explore other studies of ecosystems with nonlinear refuges.

It has been widely accepted that employing a discrete model when dealing with a limited number of populations is more rational. At a later stage, we will further investigate the dynamic characteristics of the discrete model associated with the model (3).

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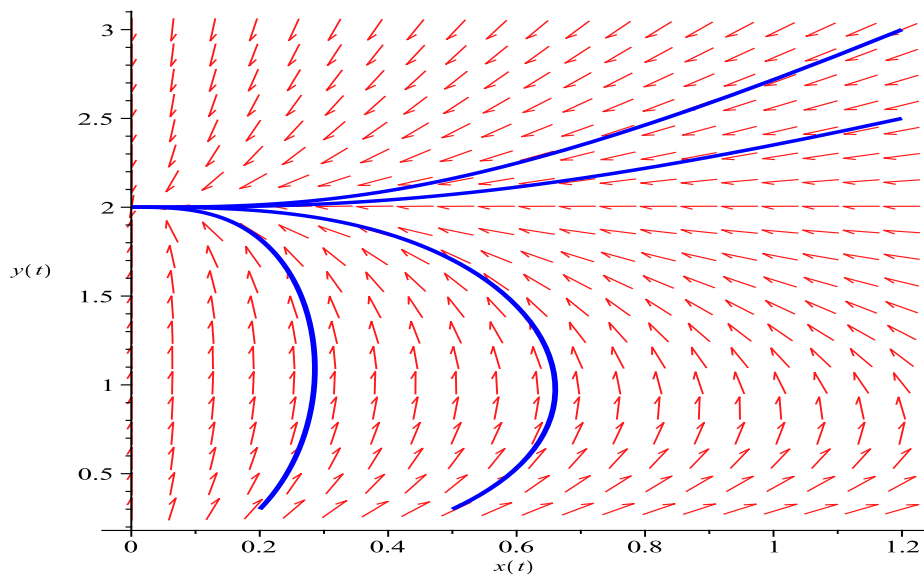


Fig. 1. Phase portrait of system (49), here  $k = \frac{1}{4}$  and the initial conditions  $(x(0), y(0)) = (1.2, 3), (1.2, 2.5), (0.5, 0.3),$  and  $(0.2, 0.3)$ , respectively.

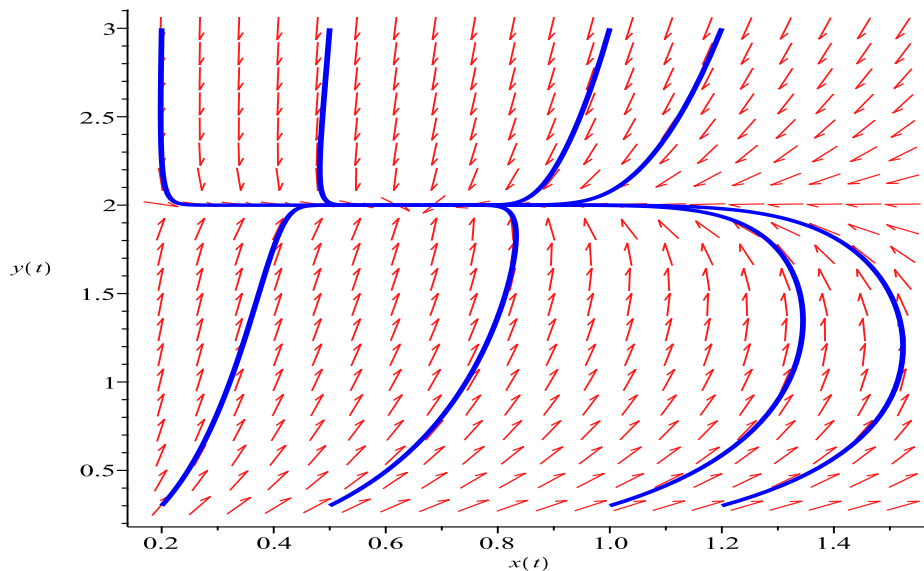


Fig. 2. Phase portrait of system (49), here  $k = \frac{7}{8}$  and the initial condition  $(x(0), y(0)) = (1.2, 3), (1, 3), (0.5, 3), (0.2, 3), (1.2, 0.3), (1, 0.3), (0.5, 0.3),$  and  $(0.2, 0.3)$ , respectively.

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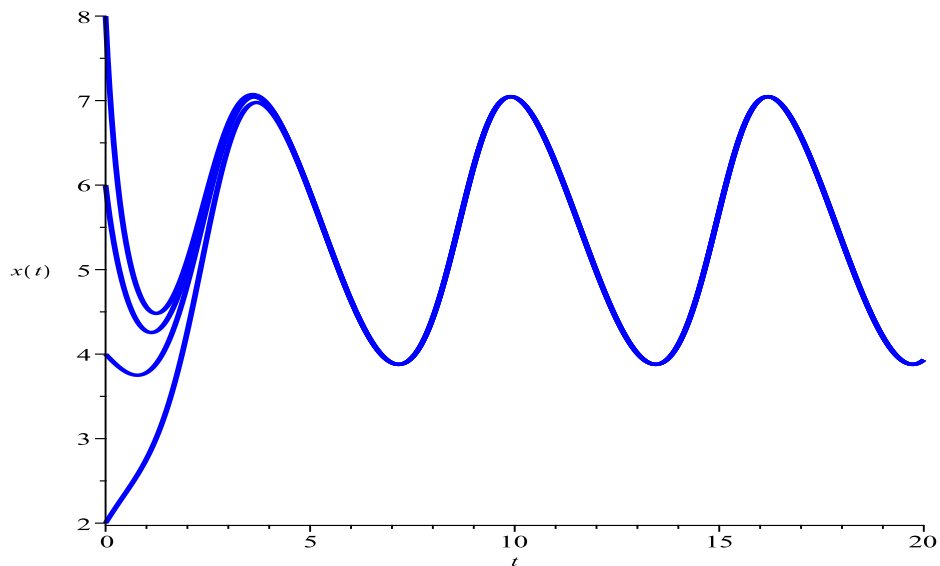


Fig. 3. Dynamic behaviors of  $x(t)$  in system (50), the initial condition  $(x(0), y(0)) = (8, 2), (6, 1.6), (0.5, 3), (4, 1.2),$  and  $(2, 1)$ , respectively.

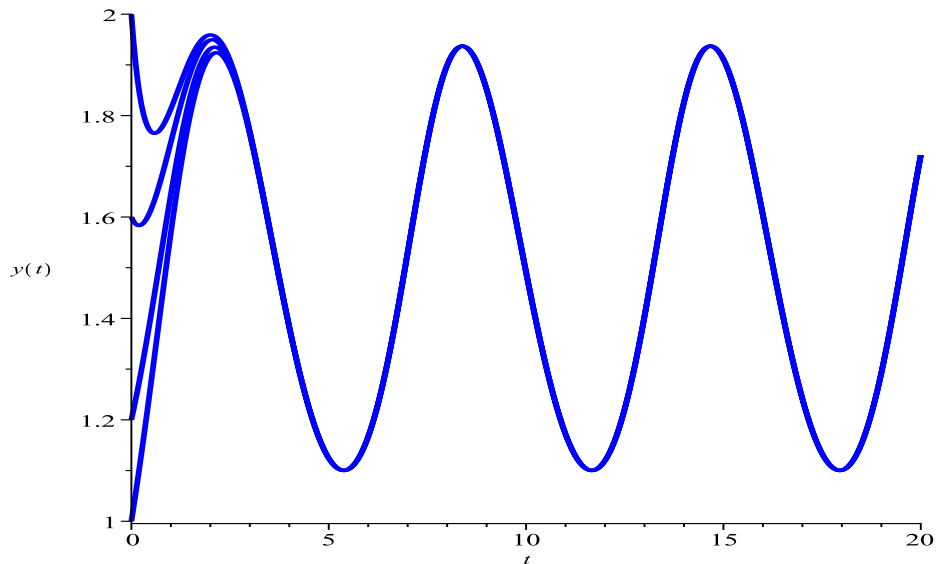


Fig. 4. Dynamic behaviors of  $y(t)$  in system (50), the initial condition  $(x(0), y(0)) = (8, 2), (6, 1.6), (0.5, 3), (4, 1.2),$  and  $(2, 1)$ , respectively.

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