Deriving Discrete Convolution Sums Through Relationships With Ramanujan-Type Eisenstein Series

Vidya Harekala Chandrashekara, Ashwath Rao Badanidiyoor and Smitha Ganesh Bhat*

***Abstract*—This study seeks to establish connections between Ramanujan-type Eisenstein series and Borweins' cubic theta functions, employing the (p, k)-parametrization method introduced by Alaca. Furthermore, the derived identities are applied to provide a novel representation for the evaluation of the discrete convolution sum $\sum_{2l+3m=n} \delta(i)\delta(j)$.**

***Index Terms*—Cubic Theta Functions, Eisenstein Series, Convolution Sum, Digital Signal Processing.**

I. Introduction

CONVOLUTION, is a fundamental mathematical operation used extensively in various fields of science and engineering. Essentially, it merges two datasets to generate a third, illustrating how one dataset influences the other's form. This entails multiplying corresponding elements of the two sets and aggregating these products, culminating in the convolution outcome.

In practical terms, convolution is often applied to vectors or matrices representing signals, images, or other forms of data. It's a crucial concept in fields such as numerical linear algebra, probability theory, numerical analysis, deep learning, and signal processing.

In signal processing, convolution plays a key role in designing finite impulse response (FIR) filters, which are used to modify or enhance signals in various applications such as audio processing, image filtering, and communication systems. By convolving an input signal with the impulse response of a filter, one can achieve desired effects like noise reduction, signal smoothing, or frequency manipulation.

Moreover, convolution serves as a foundational concept in digital signal processing (DSP), forming the theoretical backbone for understanding and implementing various signal processing algorithms and techniques. In communication systems, convolution is employed in coding and modulation schemes, enabling efficient transmission and reception of data over noisy channels.

Overall, convolution is a versatile and powerful tool that underpins many aspects of scientific and engineering disciplines, contributing to advancements in technology and enabling a wide range of practical applications.

In mathematics, particularly in the realm of number theory and special functions, researchers often utilize various tools and techniques to explore and derive properties of mathematical objects. One such area of interest involves convolution sums, which arise in the study of modular forms and related structures.

Mathematicians often leverage a diverse toolkit, incorporating Ramanujan's discriminant function, Gaussian hypergeometric series, quasi-modular forms, and Ramanujan-type Eisenstein series to dissect convolution sums. These analytical instruments offer valuable insights into the characteristics and behaviors of such sums, empowering mathematicians to unveil fresh identities and correlations.

In this article, the focus is on investigating identities involving Eisenstein series and Borweins' cubic theta functions. These identities are explored using parameters introduced by Alaca, allowing for a deeper understanding of the connections between these mathematical objects. Notably, the article presents expressions for Ramanujan-type Eisenstein series as products of cubic theta functions, offering new perspectives on their structure and properties.

Furthermore, this article applies these formulations to derive a novel representation for a specific type of discrete convolution sum, represented as $\sum_{2l+3m=n} \delta(i)\delta(j)$. This representation is valid for all positive integers l, providing a valuable contribution to the study of convolution sums and their applications.

In essence, this article enhances comprehension of convolution sums through the adept utilization of diverse mathematical tools and methodologies, yielding fresh perspectives and outcomes within the domain of number theory and special functions.

Section 2 provides essential groundwork for the research's main objectives. It likely includes fundamental definitions, theorems, and lemmas necessary to understand and approach the subsequent analysis. These preliminary results set the stage for the exploration of more advanced concepts in later sections.

In Section 3, we delve into unveiling captivating identities, characterized by their novelty, as they haven't been documented in prior literature. The intricate connections investigated are likely to involve Ramanujan-style Eisenstein series and Borweins' cubic theta functions.

In Section 4, a pioneering method for assessing discrete convolution sums is unveiled. This novel approach proposes an alternative representation, signaling a departure from conventional methods and offering a distinct framework for

Manuscript received on 2 January 2024, revised on 29 April 2024

Vidya Harekala Chandrashekara is Assistant Professor-Senior Scale at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (e-mail: vidyaashwath@gmail.com)

Ashwath Rao Badanidiyoor is Assistant Professor-Selection Grade at Department of Computer Science, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (e-mail:ashwath.rao.b@gmail.com)

Smitha Ganesh Bhat is Assistant Professor-Senior Scale at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (corresponding author to provide phone: 9844061970, e-mail: smitha.holla@manipal.edu)

comprehending and scrutinizing these sums. Such innovative perspective holds promise for uncovering fresh insights or applications within the realm of convolution sums and associated fields of inquiry.

Overall, these sections collectively contribute to the advancement of knowledge in the field by building on existing results, introducing new identities, and offering innovative perspectives on mathematical concepts.

II. Preliminaries

The genesis of the arithmetic-geometric mean iteration finds its roots in the realms of elliptic functions and theta functions. Pioneering this connection, the Borwein brothers [9], [10] unearthed a set of multidimensional theta functions, which form an essential foundation for further exploration.

$$a(q) := \sum_{r,s=-\infty}^{\infty} q^{r^2+rs+s^2}.$$
$$b(q) := \sum_{r,s=-\infty}^{\infty} \omega^{r-s} q^{r^2+rs+s^2}.$$
$$c(q) := \sum_{r,s=-\infty}^{\infty} q^{\left(r+\frac{1}{3}\right)^2+\left(r+\frac{1}{3}\right)\left(s+\frac{1}{3}\right)+\left(s+\frac{1}{3}\right)^2}.$$

for $|q| < 1$, where q represents complex numbers, and $\omega = exp(2\pi i/3)$ is the principal cube root of unity, the given expressions for two-dimensional theta functions reveal that when $q = 0$, the values become $a(q) = 1$, $b(q) = 1$, and $c(q) = 0$.

Euler's binomial theorem provides a method for expanding expressions of the form $(1+x)^n$ where n is any real number. The Borwein siblings, who are renowned mathematicians, have used this theorem as a foundational concept in their work. They have derived representations for two functions, $b(q)$ and $c(q)$, by expressing them as infinite products.

By utilizing Euler's binomial theorem, the Borwein siblings have likely found a way to express $b(q)$ and $c(q)$ as products involving terms derived from binomial expansions, potentially with coefficients that follow certain rules or patterns. These representations are likely to have mathematical significance and may offer insights into the behavior or properties of the functions $b(q)$ and $c(q)$.

Utilizing Euler's binomial theorem as a starting point, the Borwein siblings have formulated representations for both $b(q)$ and $c(q)$ in the form of infinite products, as demonstrated below:,

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} = R(q),$$
$$c(q) = \frac{3q^{\frac{1}{3}}(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

The function $a(q)$ can be expressed as follows (J.M. Borwein & P. B. Borwein [9]) and Berndt [7]:

$$a(q) = (-q;q^2)_\infty^2(q^2;q^2)_\infty(-q^3;q^6)_\infty^2(q^6;q^6)_\infty + 4q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}.$$

Or in other words,

$$a(q) = S(q) + 4T(q),$$

where

$$S(q) = (-q;q^2)_\infty(-q^3;q^6)_\infty^2(q^2;q^2)_\infty(q^6;q^6)_\infty \quad \text{and}$$
$$T(q) = q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}.$$

Besides that, they have established the fundamental relationship between $a(q)$, $b(q)$ and $c(q)$ which is a basic cubic identity given by,

$$a^3(q) = b^3(q) + c^3(q).$$

Definition II.1. *In his second notebook [15], Srinivasa Ramanujan elucidated the definitions of the Eisenstein Series $L(q)$ and $M(q)$ as outlined below:*

$$L(q) := 1 - 24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} := 1 - 24\sum_{r=1}^{\infty}\delta_1(r)q^r,$$
$$M(q) := 1 + 240\sum_{r=1}^{\infty}\frac{r^3q^r}{1-q^r} := 1 + 240\sum_{r=1}^{\infty}\delta_3(r)q^r.$$

Definition II.2. *For any complex c and d, Ramanujan[7, p.35] documented a general theta function,*

$$f(c,d) := \sum_{m=-\infty}^{\infty} c^{m(m+1)/2}d^{m(m-1)/2}$$
$$:= (-c;cd)_\infty(-d;cd)_\infty(cd;cd)_\infty,$$

where

$$(c;q)_\infty := \prod_{m=0}^{\infty}(1-cq^m), \qquad |q| < 1.$$

The special case of theta function defined by Ramanujan[7, p.35],

$$\varphi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty.$$

In their noteworthy publication, Alaca et al. [1] introduced the $(p,\ k)$ parametrization of theta functions. This parametrization holds particular significance in the formulation of the duplication and triplication principles, leading to the derivation of specific sum-to-product identities. The parameters p and k are precisely defined as follows:

$$p := p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\phi^2(q^3)}.$$
$$k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}.$$

Since $\varphi(0) = 1$, it clear that $p(0) = 0$ and $k(0) = 1$.

Lemma II.3. *[1] Concerning the previously mentioned Eisenstein series [9], [10], the expressions for $M(q)$, $M(q^l)$, $L(q) - lL(q^l)$, where $(l = 2,3,4,6,12)$, as well as $L(-q^l) - rL(q^r)$, where $l \in 1,3$ and $r \in 1,2,3$, in terms of the* *parameters p and k, are articulated as follows:*

$$
\begin{aligned}
M(q) &= (1+124p(1+p^6)+964p^2(1+p^4)+2788p^3(1\\
&\quad +p^2)+3910p^4+p^8)k^4,\\
M(q^2) &= (1+4p(1+p^6)+64p^2(1+p^4)+178p^3(1+p^2)\\
&\quad +235p^4+p^8)k^4,\\
M(q^3) &= (1+4p(1+p^6)+4p^2(1+p^4)+28p^3(1+p^2)\\
&\quad +70p^4+p^8)k^4,\\
M(q^6) &= (1+4p(1+p^6)+4p^2(1+p^4)-2p^3(1+p^2)\\
&\quad -5p^4+p^8)k^4,\\
M(q^{12}) &= (1+4p(1+p)-2p^3(1+p^2)\\
&\quad -5p^4+p^6(1+p)/4+p^8/16)k^4,\\
L(-q)-L(q) &= 3(8p+12p^2+6p^3+p^4)k^2,\\
L_{1,2}(q) &= (L(-q)-L(q))/48 = (p/2+3p^2/4\\
&\quad +3p^3/8+p^4/16)k^2,\\
L_{1,2}(q^3) &= (L(-q^3)-L(q^3))/48 = p^3(2+p)k^2/16,\\
L(-q)-2L(q^2) &= -(1-10p-12p^2-4p^3-2p^4)k^2,\\
L(q)-2L(q^2) &= -(1+14p(1+p^2)+24p^2+p^4)k^2,\\
L(q)-3L(q^3) &= -(1+8p(1+p^2)+18p^2+p^4)k^2,\\
L(q)-6L(q^6) &= -(5+22p(1+p^2)+36p^2+5p^4)k^2,\\
L(q^2)-3L(q^6) &= -2(1+2p(1+p^2)+3p^2+p^4)k^2,\\
L(q^3)-2L(q^6) &= -(1+2p(1+p^2)+p^4)k^2,\\
L(q)-4L(q^4) &= -3(1+6p+12p^2+8p^3)k^2,\\
L(q)-12L(q^{12}) &= -(11+34p+36p^2+16p^3+2p^4)k^2.
\end{aligned}
$$

Lemma II.4. *Alaca et al. [1] have derived the parametric representations of $a(q^r), b(q^r), c(q^r)$ for $r \in 1,2,4,6$, as well as $a(-q)$, $b(-q)$, $c(-q)$, expressed in terms of the parameters p and k, and are presented below.*

$$
\begin{aligned}
a(-q) &= (1-2p-2p^2)k,\\
a(q) &= (1+4p+p^2)k,\\
a(q^2) &= (1+p+p^2)k,\\
a(q^4) &= (1+p-\frac{1}{2}p^2)k,\\
a(q^6) &= \frac{(p^2+p+1+2^{1/3}((1-p)(2+p)(1+2p))^{2/3})k}{3},\\
b(-q) &= 2^{-\frac{1}{3}}((1-p)(1+2p)^4(2+p))^{\frac{1}{3}}k,\\
b(q) &= 2^{-\frac{1}{3}}((1-p)^4(1+2p)(2+p))^{\frac{1}{3}}k,\\
b(q^2) &= 2^{-2/3}((1-p)(1+2p)(2+p))^{\frac{2}{3}}k,\\
b(q^4) &= 2^{-\frac{4}{3}}((1-p)(1+2p)(2+p)^4)^{\frac{1}{3}}k,\\
c(-q) &= -2^{\frac{1}{3}}3(p(1+p))^{\frac{1}{3}}k,\\
c(q) &= 2^{-\frac{1}{3}}3(p(1+p)^4)^{\frac{1}{3}}k,\\
c(q^2) &= 2^{-\frac{2}{3}}3(p(1+p))^{\frac{2}{3}}k.\\
c(q^4) &= 2^{-\frac{4}{3}}3(p^4(1+p))^{\frac{1}{3}}k,\\
c(q^6) &= \frac{\left(p^2+p+1-2^{-2/3}((1-p)(2+p)(1+2p))^{2/3}\right)k}{3}.
\end{aligned}
$$

III. The Interplay between Ramanujan's Eisenstein Series and Cubic Theta Functions: A Comprehensive Exploration of their Mathematical Connections and Applications

In Ramanujan's notebook [15], he documented intriguing series involving L, M, and N, which led to a multitude of notable identities for infinite series encompassing theta functions. Computational methods, as detailed by Xia et al. [24], were utilized to unveil elegant mathematical identities involving both Eisenstein series and cubic theta functions. These identities, primarily in the form $L(q)-rL(q^r)$, where r takes on values from the set {2, 3, 4, 6, 12}, were further expanded upon by Shruti and Srivatsakumar B.R. [16], who also evaluated convolution sums.

More recently, Vidya H. C. and Ashwath Rao B. [20], as well as Vidya H. C. and Smitha G. Bhat [21], and Smitha G. Bhat et al. [18], contributed additional identities, particularly those involving $L(-q^l)-L(q^l)$, where $l \in \{1,3\}$. Vidya H. C. and Ashwath Rao B. [12] extended these investigations to deduce relationships among theta functions.

In this paper, we present specific identities connecting Ramanujan-type Eisenstein series and cubic theta functions, with a focus on Eisenstein series $M(q^n)$, where n ranges over {1, 2, 3, 6, 12}. Notably, these connections were established without the aid of computer assistance. Furthermore, we leverage these findings to evaluate convolution sums.

Theorem III.1. *The connection between an infinite series and theta functions is as follows:*

$$
(i)\ 1-12\sum_{r=1}^{\infty}\left[\frac{3rq^{2r}}{1-q^{2r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{9rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\right] = a(q)a(-q). \tag{1}
$$

$$
(ii)\ 1+\frac{3}{2}\sum_{r=1}^{\infty}\left[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{2rq^{r}}{1-q^{r}}-\frac{2rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}+\frac{2rq^{4r}}{1-q^{4r}}-\frac{18rq^{6r}}{1-q^{6r}}-\frac{6rq^{12r}}{1-q^{12r}}\right] = \frac{b(q^4)b(q^2)}{b(q)}. \tag{2}
$$

Proof: Let us persume that,

$$
\begin{aligned}
&C_1L_{1,2}(q^3)+C_2[L(q)-6L(q^6)]+C_3[L(q^2)-3L(q^6)]\\
&+C_4[L(q^3)-2L(q^6)]+C_5[L(q^4)-3L(q^{12})]\\
&= a(q)a(-q).
\end{aligned} \tag{3}
$$

The equation above undergoes transformation through (p,k) parametrization using Lemma II.3. This leads to the derivation of a system of non-homogeneous linear equations, where the coefficients of terms involving k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 in the left-hand side are equated with their corresponding terms in the right-hand side. These equations must then be solved to find the unknown values.

$$\begin{pmatrix} 0 & -5 & -2 & -1 & -2 \\ 0 & -22 & -4 & -2 & -4 \\ 0 & -36 & -6 & 0 & 0 \\ \frac{1}{8} & -22 & -4 & -2 & 2 \\ \frac{1}{16} & -5 & -2 & -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -9 \\ -10 \\ -2 \end{pmatrix}.$$

On solving the above system, we get,

$$C_1 = 0,\ C_2 = 0,\ C_3 = \frac{3}{2}, C_4 = 0 \text{ and } C_5 = -2.$$

Upon replacing the previously mentioned statistics into (3) and subsequently streamlining the process with the help of Definition II.1, we arrive at equation (1). Likewise, employing the same approach, we deduce the following identities. Altering the right-hand side of (1) and subsequently using (3), results in equations (i).

$$(i)\ -9L_{1,2}(q^3) - \frac{1}{8}[L(q) - 6L(q^6)] + \frac{1}{8}[L(q^2) - 3L(q^6)] - \frac{3}{8}[L(q^3) - 2L(q^6)] - \frac{1}{8}[L(q^4) - 3L(q^{12})] = \frac{b(q^4)b(q^2)}{b(q)}.$$

After applying the definition of Eisenstein series and simplifying, we arrive at equations (2). ■

Theorem III.2. *The connection between an infinite series and theta functions is as follows:*

$$(i)\ 1 - 12\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r} - \frac{3r(-q)^{3r}}{1-(-q)^{3r}} + \frac{rq^r}{1-q^r} - \frac{3rq^{2r}}{1-q^{2r}} - \frac{3rq^{3r}}{1-q^{3r}} + \frac{9rq^{6r}}{1-q^{6r}}\right] = a(q)a(-q). \quad (4)$$

$$(ii)\ 1 - 3\sum_{r=1}^{\infty}\left[\frac{8rq^r}{1-q^r} - \frac{2rq^{2r}}{1-q^{2r}} - \frac{9rq^{3r}}{1-q^{3r}} + \frac{18rq^{6r}}{1-q^{6r}}\right] = b(q)b(q^2). \quad (5)$$

$$(iii)\ 1 - \frac{3}{4}\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r} - \frac{9r(-q)^{3r}}{1-(-q)^{3r}} - \frac{3rq^r}{1-q^r} - \frac{2rq^{2r}}{1-q^{2r}} - \frac{9rq^{3r}}{1-q^{3r}} + \frac{54rq^{6r}}{1-q^{6r}}\right] = \frac{b(q^4)b(q^2)}{b(q)}. \quad (6)$$

Proof: Let us persume that,

$$C_1L_{1,2}(q) + C_2L_{1,2}(q^3) + C_3[L(q) - 6L(q^6)] + C_4[L(q^2) - 3L(q^6)] + C_5[L(q^3) - 2L(q^6)] = a(q)a(-q). \quad (7)$$

Utilizing the (p,k) parametrization according to Lemma II.3, the given equation undergoes a transformation. This transformation results in the formulation of a system of non-homogeneous linear equations, where the coefficients of terms containing k^2, pk^2, p^2k^2, p^3k^2, and p^4k^2 on the left-hand side are equated with their respective counterparts on the right-hand side. The subsequent task involves solving these equations to determine the unknown values.

$$\begin{pmatrix} 0 & 0 & -5 & -2 & -1 \\ \frac{1}{2} & 0 & -22 & -4 & -2 \\ \frac{3}{4} & 0 & -36 & -6 & 0 \\ \frac{3}{8} & \frac{1}{8} & -22 & -4 & -2 \\ \frac{1}{16} & \frac{1}{16} & -5 & -2 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -9 \\ -10 \\ -2 \end{pmatrix}.$$

On solving the above system, we get,

$$C_1 = 24,\ C_2 = -72,\ C_3 = 1,\ C_4 = -\frac{3}{2} \text{ and } C_5 = -3.$$

Upon replacing the previously mentioned statistics into (7) and subsequently streamlining the process with the help of Definition II.1, we arrive at equation (4). Similarly, employing a parallel methodology, we derive the ensuing identities. Modifying the right-hand side of (4) and subsequently applying (7) yields the expressions denoted as equations (i) and (ii).

$$(i)\ \frac{1}{8}[L(q) - 6L(q^6)] - \frac{1}{4}[L(q^2) - 3L(q^6)] - \frac{9}{8}[L(q^3) - 2L(q^6)] = b(q)b(q^2).$$

$$(ii)\ \frac{3}{2}L_{1,2}(q) - \frac{27}{2}L_{1,2}(q^3) - \frac{1}{16}[L(q) - 6L(q^6)] - \frac{1}{16}[L(q^2) - 3L(q^6)] - \frac{9}{16}[L(q^3) - 2L(q^6)] = \frac{b(q^4)b(q^2)}{b(q)}.$$

Upon simplification utilizing the Eisenstein series definition, we arrive at equations (5) through (6). ■

Theorem III.3. *The relationship linking an infinite series and theta functions can be expressed as:*

$$(i)\ 1 + \frac{3}{2}\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r} + \frac{3rq^r}{1-q^r} - \frac{8rq^{2r}}{1-q^{2r}} + \frac{6rq^{4r}}{1-q^{4r}} - \frac{18rq^{12r}}{1-q^{12r}}\right] = \frac{b^2(q^4)b(q^2)}{b(q)}. \quad (8)$$

$$(ii)\ 1 + 4\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r} + \frac{rq^{2r}}{1-q^{2r}} - \frac{6rq^{3r}}{1-q^{3r}} - \frac{3rq^{6r}}{1-q^{6r}}\right] = \frac{b^2(q)c^2(-q)}{3.2^{\frac{4}{3}}b(q^2)c(q^2)}. \quad (9)$$

$$(iii)\ 1 + \frac{4}{3}\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r} + \frac{2rq^r}{1-q^r} - \frac{9rq^{2r}}{1-q^{2r}} + \frac{8rq^{4r}}{1-q^{4r}} + \frac{3rq^{6r}}{1-q^{6r}} - \frac{24rq^{12r}}{1-q^{12r}}\right] = \frac{c^2(-q)c^2(q)}{2^{\frac{4}{3}}c^2(q^2)}. \quad (10)$$

Proof: Let us persume that,

$$C_1[L(-q)-2L(q^2)]+C_2[L(q)-6L(q^6)]+C_3[L(q^2)-3L(q^6)]+C_4[L(q^3)-2L(q^6)]+C_5[L(q^4)-3L(q^{12})]=\frac{b^2(q^4)b(q^2)}{b(q)}. \tag{11}$$

The equation above undergoes a transformation through the (p,k) parametrization, utilizing Lemma II.3. This transformation results in a system of non-homogeneous linear equations, where the coefficients of terms containing k^2, pk^2, p^2k^2, p^3k^2, and p^4k^2 on the left-hand side are set equal to their respective terms on the right-hand side. Subsequently, these equations need to be solved to determine the unknown values.

$$\begin{pmatrix} -1 & -5 & -2 & -1 & -2 \\ 10 & -22 & -4 & -2 & -4 \\ 12 & -36 & -6 & 0 & 0 \\ 4 & -22 & -4 & -2 & 2 \\ 2 & -5 & -2 & -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix}=\begin{pmatrix} 1 \\ \frac{7}{2} \\ \frac{15}{4} \\ \frac{13}{8} \\ \frac{1}{4} \end{pmatrix}.$$

On solving the above system, we get,

$$C_1=-1,\ C_2=0,\ C_3=-\frac{5}{2},\ C_4=3 \text{ and } C_5=0.$$

Upon replacing the previously mentioned statistics into (11) and subsequently streamlining the process with the help of Definition II.1, we arrive at equation (8). Similarly, employing a parallel methodology, we derive the ensuing identities. Modifying the right-hand side of (8) and subsequently applying (11) yields the expressions denoted as equations (i) and (ii).

$$(i)\quad -[L(-q)-2L(q^2)]-\frac{5}{2}[L(q^2)-3L(q^6)]+3[L(q^3)-2L(q^6)]=\frac{b^2(q)c^2(-q)}{2^{\frac{4}{3}}b(q^2)c(q^2)}.$$

$$(ii)\quad -\frac{1}{9}[L(-q)-2L(q^2)]-\frac{1}{9}[L(q)-6L(q^6)]+\frac{5}{18}[L(q^2)-3L(q^6)]-\frac{4}{9}[L(q^4)-3L(q^{12})]=\frac{c^2(q)c^2(-q)}{2^{\frac{4}{3}}c^2(q^2)}.$$

Upon simplification through the utilization of the Eisenstein series definition, equations (9) through (10) are obtained. ■

Theorem III.4. *The relation amongst an infinite series and theta functions holds:*

$$(i)\quad \Big(1-u-4v\Big)+48\Big(u+v\Big)\sum_{r=1}^{\infty}\left[\frac{r^3q^r}{1-q^r}\right]-3\Big(1+64u\Big)\sum_{r=1}^{\infty}\left[\frac{r^3q^{2r}}{1-q^{2r}}\right]-864v\sum_{r=1}^{\infty}\left[\frac{r^3q^{3r}}{1-q^{3r}}\right]+243\sum_{r=1}^{\infty}\left[\frac{r^3q^{6r}}{1-q^{6r}}\right]+u\Big[-1-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2+v\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a(q^2)b^3(q^2). \tag{12}$$

$$(ii)\quad \Big(1-u-4v\Big)-3\Big(1+16u+32v\Big)\sum_{r=1}^{\infty}\left[\frac{r^3q^r}{1-q^r}\right]-192u\sum_{r=1}^{\infty}\left[\frac{r^3q^{2r}}{1-q^{2r}}\right]+3(81-288v)\sum_{r=1}^{\infty}\left[\frac{r^3q^{3r}}{1-q^{3r}}\right]+u\Big[-1-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2+v\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a(q)b^3(q). \tag{13}$$

$$(iii)\quad \Big(1-u-4v\Big)+6\Big(3-8u-16v\Big)\sum_{r=1}^{\infty}\left[\frac{r^3q^r}{1-q^r}\right]-48(1+4u)\sum_{r=1}^{\infty}\left[\frac{r^3q^{2r}}{1-q^{2r}}\right]-6(27+144v)\sum_{r=1}^{\infty}\left[\frac{r^3q^{3r}}{1-q^{3r}}\right]+432\sum_{r=1}^{\infty}\left[\frac{r^3q^{6r}}{1-q^{6r}}\right]+u\Big[-1-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2+v\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a^3(q)a(q^2). \tag{14}$$

$$(iv)\quad \Big(1-u-4v\Big)+24\Big(1-2u-2v\Big)\sum_{r=1}^{\infty}\left[\frac{r^3q^r}{1-q^r}\right]-192u\sum_{r=1}^{\infty}\left[\frac{r^3q^{2r}}{1-q^{2r}}\right]+216(1-4v)\sum_{r=1}^{\infty}\left[\frac{r^3q^{3r}}{1-q^{3r}}\right]$$

$$+ u\Big[-1-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2$$
$$+ v\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2$$
$$= (3a(q^3)-2b(q))^4. \tag{15}$$

$$(v)\ \Big(-2-u-4v\Big)-24\Big(1-2u-4v\Big)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big] + 192(1-u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{2r}}{1-q^{2r}}\Big]-216(3+4v) \sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+u\Big[-1-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r} -\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2+v\Big[-2 -24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2+3\Big[1 -24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2=\frac{b^8(q)}{b^4(q^2)}. \tag{16}$$

$$(vi)\ \Big(-\frac{1}{1728}-u-v\Big)+48\Big(u+2v\Big)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big] +\frac{1}{18}(1+3456u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{2r}}{1-q^{2r}}\Big]-\frac{1}{12}(1-10368v) \sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+\frac{1}{6}\sum_{r=1}^{\infty}\Big[\frac{rq^{6r}}{1-q^{6r}}\Big]+u\Big[-1 -24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2+v\Big[-2-24 \sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2+\frac{1}{1728}\Big[1 -24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2=\frac{c^8(q^2)}{81c^4(q)}. \tag{17}$$

$$(vii)\ \Big(\frac{1294}{945}-u-v\Big)+8\Big(1-6u-12v\Big)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big] -\frac{960}{135}(5+27u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{2r}}{1-q^{2r}}\Big]-\frac{8}{3}(31+324v) \sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+\frac{512}{15}\sum_{r=1}^{\infty}\Big[\frac{rq^{6r}}{1-q^{6r}}\Big]+u\Big[-1-24 \sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2+v\Big[-2-24 \sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2+\frac{1}{27}\Big[1-24 \sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big]^2=\frac{c^8(q^2)}{81c^4(q)}. \tag{18}$$

Proof:

$$C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)+C_5M(q^{12}) + C_6\{L(q)-2L(q^2)\}^2+C_7\{L(q)-3L(q^3)\}^2 + C_8\{3L(q^3)-2L(q^2)\}^2=a(q^2)b^3(q^2). \tag{19}$$

We utilize the (p,k) parametrization as described in Lemma II.3 to modify the given equation. This transformation results in a set of non-homogeneous linear equations. By equating coefficients of terms involving various powers of k and p, such as k^4, pk^4, p^2k^4, and so on, on both sides of the equation, we generate these equations. Solving for the unknown values becomes necessary to find solutions for the system.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1\\ 124 & 4 & 4 & 4 & 4 & 28 & 64 & 4\\ 964 & 64 & 4 & 4 & 4 & 244 & 400 & 28\\ 2788 & 178 & 28 & -2 & -2 & 700 & 1216 & 52\\ 3910 & 235 & 70 & -5 & -5 & 970 & 1816 & 154\\ 2788 & 178 & 28 & -2 & -2 & 700 & 1216 & 52\\ 964 & 64 & 4 & 4 & \frac{1}{4} & 244 & 400 & 28\\ 124 & 4 & 4 & 5 & \frac{1}{4} & 28 & 64 & 4\\ 1 & 1 & 1 & 1 & \frac{1}{16} & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8 \end{pmatrix} = \begin{pmatrix} 1\\ 4\\ \frac{13}{4}\\ -\frac{17}{4}\\ -8\\ -\frac{17}{4}\\ \frac{13}{4}\\ 4\\ 1 \end{pmatrix}.$$

We note that, the system results in an infinitely many solutions,

$$C_1=-\frac{u}{5}-\frac{2v}{5},\ C_2=-\frac{1}{80}-\frac{4u}{5},\ C_3=-\frac{18v}{5},$$
$$C_4=\frac{81}{80},\ C_5=0,\ C_6=u,\ C_7=v,\ C_8=0.$$

Substituting these values in (19) yields, (12). Similarly, altering the right-hand side of (12) and subsequently using (19), results in equations (i) to (vi).

$$(i)\ \Big(-\frac{1}{80}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\frac{4u}{5}M(q^2)+\Big(\frac{81}{80}-\frac{18v}{5}\Big)M(q^3)+u\{L(q)-2L(q^2)\}^2+v\{L(q)-3L(q^3)\}^2=a(q)b^3(q).$$

$$(ii)\ \Big(\frac{3}{40}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\Big(\frac{1}{5}-\frac{4u}{5}\Big)M(q^2)-\Big(\frac{27}{40}+\frac{18v}{5}\Big)M(q^3)+\frac{9}{5}M(q^6)+u\{L(q)-2L(q^2)\}^2+v\{L(q)-3L(q^3)\}^2=a^3(q)a(q^2).$$

$$(iii)\ \Big(\frac{1}{10}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\frac{4u}{5}M(q^2)+\Big(\frac{9}{10}-\frac{18v}{5}\Big)M(q^3)+u\{L(q)-2L(q^2)\}^2+v\{L(q)-3L(q^3)\}^2=(3a(q^3)-2b(q))^4.$$

$$(iv)\ \Big(\frac{1}{10}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)+\Big(\frac{4}{5}-\frac{4u}{5}\Big)M(q^2)-\Big(\frac{27}{10}+\frac{18v}{5}\Big)M(q^3)+u\{L(q)-2L(q^2)\}^2+v\{L(q)-3L(q^3)\}^2+3\{3L(q^3)-2L(q^2)\}^2=\frac{b^8(q)}{b^4(q^2)}.$$

$$(v)\ \Big(-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\Big(\frac{1}{4320}+\frac{4u}{5}\Big)M(q^2)+\Big(\frac{1}{2880}-\frac{18v}{5}\Big)M(q^3)-\frac{1}{1440}M(q^6)+u\{L(q)-2L(q^2)\}^2+v\{L(q)-3L(q^3)\}^2+\frac{1}{1728}\{3L(q^3)-2L(q^2)\}^2=\frac{c^8(q^2)}{81c^4(q)}.$$

$$(vi)\ \Big(\frac{1}{30}-\frac{u}{5}-\frac{2v}{5}\Big)\Big(\frac{1}{10}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\Big(\frac{4}{27}+\frac{4u}{5}\Big)M(q^2)-\Big(\frac{31}{90}+\frac{18v}{5}\Big)M(q^3)+\frac{64}{45}M(q^6)+u\{L(q)-2L(q^2)\}^2+v\{L(q)-3L(q^3)\}^2+\frac{1}{27}\{3L(q^3)-2L(q^2)\}^2=\frac{c^8(q)}{81c^4(q^2)}.$$

Upon employing the definition of Eisenstein series and simplifying, equations (13) to (18) emerge. ■

Theorem III.5. *The relation amongst an infinite series and theta functions holds:*

$$(i)\ \Big(1-4u\Big)-96u\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big]-3\sum_{r=1}^{\infty}\Big[\frac{r^3q^{2r}}{1-q^{2r}}\Big]-864u\sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+243\sum_{r=1}^{\infty}\Big[\frac{r^3q^{6r}}{1-q^{6r}}\Big]+u\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a(q^2)b^3(q^2).\quad(20)$$

$$(ii)\ \Big(1-4u\Big)-3(1+32u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big]+27(9-32u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+u\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a(q)b^3(q).\quad(21)$$

$$(iii)\ \Big(1-4u\Big)+6(3-16u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big]-48\sum_{r=1}^{\infty}\Big[\frac{r^3q^{2r}}{1-q^{2r}}\Big]-27(3+16u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+432\sum_{r=1}^{\infty}\Big[\frac{r^3q^{6r}}{1-q^{6r}}\Big]+u\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a^3(q)a(q^2).\quad(22)$$

$$(iv)\ \Big(1-4u\Big)+24(1-4u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big]+216(1-10u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+u\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=(3a(q^3)-2b(q))^4.\quad(23)$$

$$(v)\ \Big(-2-4u\Big)-24(1+4u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big]+192\sum_{r=1}^{\infty}\Big[\frac{r^3q^{2r}}{1-q^{2r}}\Big]-216(3+4u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]+u\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=\frac{b^8(q)}{b^4(q^2)}.\quad(24)$$

$$(vi)\ \Big(-\frac{1}{1728}-4u\Big)-96u\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big]-\frac{1}{8}\sum_{r=1}^{\infty}\Big[\frac{r^3q^{2r}}{1-q^{2r}}\Big]+\frac{1}{12}(1-10368u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^{3r}}{1-q^{3r}}\Big]-\frac{1}{6}\sum_{r=1}^{\infty}\Big[\frac{r^3q^{6r}}{1-q^{6r}}\Big]+u\Big[-2-24\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2+\frac{1}{1728}\Big[-1-24\sum_{r=1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big]^2=\frac{c^8(q^2)}{81c^4(q)}.\quad(25)$$

$$(vii)\ \Big(-\frac{1}{27}-4u\Big)-8(1-12u)\sum_{r=1}^{\infty}\Big[\frac{r^3q^r}{1-q^r}\Big]$$

$$
\begin{aligned}
&-\frac{320}{9}\sum_{r=1}^{\infty}\left[\frac{r^3q^{2r}}{1-q^{2r}}\right]-\frac{8}{3}(31+324u)\sum_{r=1}^{\infty}\left[\frac{r^3q^{3r}}{1-q^{3r}}\right]\\
&+\frac{3072}{9}\sum_{r=1}^{\infty}\left[\frac{r^3q^{6r}}{1-q^{6r}}\right]+u\Bigg[-2-24\sum_{r=1}^{\infty}\Bigg[\frac{rq^r}{1-q^r}\\
&-\frac{3rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2+\frac{1}{27}\Bigg[-1-24\sum_{r=1}^{\infty}\Bigg[\frac{2rq^{2r}}{1-q^{2r}}\\
&-\frac{3rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2=\frac{c^8(q)}{81c^4(q^2)}. \qquad (26)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
&C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)+C_5M(q^{12})\\
&+C_6\{L(q)-3L(q^3)\}^2+C_7\{L(q)-4L(q^4)\}^2\\
&+C_8\{3L(q^3)-2L(q^2)\}^2=a(q^2)b^3(q^2). \qquad (27)
\end{aligned}
$$

By employing the (p,k) parametrization according to Lemma II.3, we modify the given equation. This results in a set of non-homogeneous linear equations, achieved by equating coefficients of terms involving $k^4, pk^4, p^2k^4, p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$, and p^8k^4 on the left-hand side with their corresponding terms on the right-hand side. The subsequent step involves solving for the unknown values.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 4 & 9 & 1\\
124 & 4 & 4 & 4 & 4 & 64 & 108 & 4\\
964 & 64 & 4 & 4 & 4 & 400 & 540 & 28\\
2788 & 178 & 28 & -2 & -2 & 1216 & 1440 & 52\\
3910 & 235 & 70 & -5 & -5 & 1816 & 2160 & 154\\
2788 & 178 & 28 & -2 & -2 & 1216 & 1728 & 52\\
964 & 64 & 4 & 4 & \frac{1}{4} & 400 & 576 & 28\\
124 & 4 & 4 & 5 & \frac{1}{4} & 64 & 0 & 4\\
1 & 1 & 1 & 1 & \frac{1}{16} & 4 & 0 & 1
\end{pmatrix}
\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8 \end{pmatrix}
=
\begin{pmatrix} 1\\ 4\\ \frac{13}{4}\\ -\frac{17}{4}\\ -8\\ -\frac{17}{4}\\ \frac{13}{4}\\ 4\\ 1 \end{pmatrix}.
$$

It's worth observing that the system yields an infinite set of solutions.

$$
\begin{aligned}
&C_1=-\frac{2u}{5},\ C_2=-\frac{1}{80},\ C_3=-\frac{18u}{5},\ C_4=\frac{81}{80},\\
&\qquad C_5=0,\ C_6=u,\ C_7=0,\ C_8=0.
\end{aligned}
$$

Substituting these values in (27) yields, (20). Similarly, altering the right-hand side of (20) and subsequently using (27), results in equations (i) to (vi).

$$
\begin{aligned}
(i)\ &\left(-\frac{1}{80}-\frac{2u}{5}\right)M(q)+\left(\frac{81}{80}-\frac{18}{5}\right)M(q^3)+u\{L(q)\\
&-3L(q^3)\}^2=a(q)b^3(q). \qquad (28)\\
(ii)\ &\left(\frac{3}{40}-\frac{2u}{5}\right)M(q)-\frac{1}{5}M(q^2)-\left(\frac{27}{40}+\frac{18u}{5}\right)M(q^3)\\
&+\frac{9}{5}M(q^6)+u\{L(q)-3L(q^3)\}^2=a^3(q)a(q^2). \qquad (29)\\
(iii)\ &\left(\frac{1}{10}-\frac{2u}{5}\right)M(q)+\left(\frac{9}{10}-\frac{18u}{5}\right)M(q^3)+u\{L(q)\\
&-3L(q^3)\}^2=(3a(q^3)-2b(q))^4. \qquad (30)\\
(iv)\ &\left(-\frac{1}{10}-\frac{2u}{5}\right)M(q)+\frac{4}{5}M(q^2)-\left(\frac{27}{10}+\frac{18u}{5}\right)M(q^3)\\
&+u\{L(q)-3L(q^3)\}^2+3\{3L(q^3)-2L(q^2)\}^2\\
&=a(q^2)b^3(q^2). \qquad (31)\\
(v)\ &-\frac{2u}{5}M(q)-\left(\frac{1}{4320}\right)M(q^2)+\left(\frac{1}{2880}-\frac{18u}{5}\right)M(q^3)\\
&-\frac{1}{1440}M(q^6)+u\{L(q)-3L(q^3)\}^2+\frac{1}{1728}\{3L(q^3)\\
&-2L(q^2)\}^2=\frac{c^8(q^2)}{81c^4(q)}. \qquad (32)\\
(vi)\ &\left(\frac{1}{30}-\frac{2u}{5}\right)M(q)-\frac{4}{27}M(q^2)-\left(\frac{31}{90}+\frac{18u}{5}\right)M(q^3)\\
&+\frac{64}{45}M(q^6)+u\{L(q)-3L(q^3)\}^2+\frac{1}{27}\{3L(q^3)\\
&-2L(q^2)\}^2=\frac{c^8(q)}{81c^4(q^2)}. \qquad (33)
\end{aligned}
$$

By applying the definition of Eisenstein series and simplifying, we derive equations (21) to (26). ■

IV. Evaluation of Convolution sum $\sum_{r=2i+3j}\delta(i)\delta(j)$

Consider the set of natural numbers denoted as $\mathbb{N}$. For any $k, r \in \mathbb{N}$, we establish the following definition:

$$\delta_k(r)=\sum_{d/r}d^k.$$

here, d traverses the non-negative integral divisors of r.
For $i,\ j,\ r \in \mathbb{N}$ with $i \leq j$, the convolution sum is expressed as:

$$W_{i,j}(r):=\sum_{il+jk=r}\delta(l)\delta(k).$$

Alaca et al.[1], [2], [3], [4], [5], [6], H. C. Vidya, and B. R. Srivtasa Kumar [19], Williams et al.[22], [23], as well as E. X. W. Xia and O. X. M. Yao [24], have extensively and explicitly computed the convolution $\sum_{li+kj=r}\delta(i)\delta(j)$ for a range of i and j values across all r.
The assertions made by J. W. L. Glaisher [11] provide support and validation for our proof.

$$L^2(q)=1+\sum_{r=1}^{\infty}(240\delta_3(r)-288r\delta_1(r))q^r. \qquad (34)$$

Theorem IV.1. *For every $r \in \mathbb{N}$ and $u \in \mathbb{R}-0$, the following holds:*

$$\sum_{2i+3j=r} \delta(i)\delta(j) = \left(\frac{1}{24}-\frac{r}{6}\right)\delta_1\left(\frac{r}{2}\right) + \left(\frac{1}{24}-\frac{3r}{8}\right)\delta_1\left(\frac{r}{3}\right) - \left(\frac{7}{3456}\right)\delta_3(r) + \left(\frac{65}{432}\right)\delta_3\left(\frac{r}{2}\right) + \left(\frac{37}{128}\right)\delta_3\left(\frac{r}{3}\right) - \left(\frac{1}{48}\right)\delta_3\left(\frac{r}{6}\right) + \frac{1}{20736}\left(B(r)-A(r)\right), \tag{35}$$

where $1+\sum_{r=1}^{\infty}A(r)q^r = \Big(S(q^2)+4T(q^2)\Big)R^3(q^2)$ *and*
$1+\sum_{r=1}^{\infty}B(r)q^r = \Big(S(q)+4T(q)\Big)^3\Big(S(q^2)+4T(q^2)\Big)$

Proof: Consider (29).

$$\left(\frac{3}{40}-\frac{2u}{5}\right)M(q) - \frac{1}{5}M(q^2) - \left(\frac{27}{40}+\frac{18u}{5}\right)M(q^3) + \frac{9}{5}M(q^6) + u\{L(q)-3L(q^3)\}^2 = a^3(q)a(q^2).$$

which yeilds,

$$\left(\frac{3}{40}-\frac{2u}{5}\right)\left[1+\sum_{r=1}^{\infty}240\delta_3(r)q^r\right] - \frac{1}{5}\left[1+\sum_{r=1}^{\infty}240\delta_3\left(\frac{r}{2}\right)q^r\right] - \left(\frac{27}{40}+\frac{18u}{5}\right)\left[1+\sum_{r=1}^{\infty}240\delta_3\left(\frac{r}{3}\right)q^r\right] + \frac{9}{5}\left[1+\sum_{r=1}^{\infty}240\delta_3\left(\frac{r}{6}\right)q^r\right] + \sum_{r=1}^{\infty}\left[240\delta_3(r) - 288ru\delta_1(r) + 2160u\delta_3\left(\frac{r}{3}\right) - 2592ru\delta_1\left(\frac{r}{3}\right)\right]q^r + 4u + 144u\sum_{r=1}^{\infty}\delta_1(r)q^r + 144u\sum_{r=1}^{\infty}\delta_1\left(\frac{r}{3}\right)q^r - 3456u\sum_{i+3j=r}\delta(i)\delta(j)q^r = a^3(q)a(q^2).$$

On simplifying,

$$\sum_{i+3j=r}\delta(i)\delta(j) = \sum_{r=1}^{\infty}\left[\left(\frac{1}{24}-\frac{r}{12}\right)\delta_1(r) + \left(\frac{1}{24}-\frac{3r}{4}\right)\delta_1\left(\frac{r}{3}\right) + \left(\frac{1}{1924}+\frac{1}{24}\right)\delta_3(r) - \left(\frac{1}{72u}\right)\delta_3\left(\frac{r}{2}\right) - \left(\frac{3}{64u}-\frac{3}{8}\right)\delta_3\left(\frac{r}{3}\right) + \left(\frac{1}{8u}\right)\delta_3\left(\frac{r}{6}\right) - \left(\frac{1}{3456u}\right)B(r)\right]$$

Utilizing Definition II.1 on equation (31) and rearranging, we derive:

$$- \left[\frac{1}{10}+\frac{2u}{5}\right]\left[1+240\sum_{r=1}^{\infty}\delta_3(r)q^r\right] + \left[\frac{4}{5}\right]\left[1+240\sum_{r=1}^{\infty}\delta_3\left(\frac{r}{2}\right)q^r\right] - \left[\frac{27}{10}+\frac{18u}{5}\right]\left[1+240\sum_{r=1}^{\infty}\delta_3\left(\frac{r}{3}\right)q^r\right] + 4u + u\left[1+\sum_{r=1}^{\infty}\left[240\delta_3(r) - 288r\delta_1(r) + 2160\delta_3\left(\frac{r}{3}\right) - 2592r\delta_3\left(\frac{r}{3}\right)\right]q^r + 144u\sum_{r=1}^{\infty}\delta_1(r)q^r + 144u\sum_{r=1}^{\infty}\delta_1\left(\frac{r}{3}\right)q^r - 3456u\sum_{r=i+3j}\delta(i)\delta(j)q^r + 3 + \sum_{r=1}^{\infty}\left[6480\delta_3\left(\frac{r}{3}\right) - 7776r\delta_1\left(\frac{r}{3}\right) + 2880\delta_3\left(\frac{r}{2}\right) - 3456r\delta_1\left(\frac{r}{2}\right)\right]q^r + 864\sum_{r=1}^{\infty}\delta_1\left(\frac{r}{3}\right)q^r + 864\delta_1\left(\frac{r}{2}\right)q^r - 20736\sum_{r=2i+3j}\delta(i)\delta(j)q^r = 1+\sum_{r=1}^{\infty}A(r)q^r,$$

Rearranging,

$$1+\sum_{r=1}^{\infty}\left[\Big(144u-288ru\Big)\delta_1(r)\right]q^r + \sum_{r=1}^{\infty}\left[\Big(864 - 3456r\Big)\delta_1\left(\frac{r}{2}\right)\right]q^r + \sum_{r=1}^{\infty}\left[\Big(-2592ru+144u - 7776r+864\Big)\delta_1\left(\frac{r}{3}\right)\right]q^r + \sum_{r=1}^{\infty}\left[\Big(144u-24\Big)\delta_3(r) + 3072\delta_3\left(\frac{r}{2}\right)\right]q^r + \sum_{r=1}^{\infty}\left[\Big(1296u+5832\Big)\delta_3\left(\frac{r}{3}\right)\right]q^r - \sum_{r=1}^{\infty}\left[\Big(144u-288ru\Big)\delta_1(r) + \Big(144u-2592ru\Big)\delta_1\left(\frac{r}{3}\right) + \Big(18+144u\Big)\delta_3(r)\right]q^r + \sum_{r=1}^{\infty}\left[-48\delta_3\left(\frac{r}{2}\right) + \Big(1296u - 162\Big)\delta_3\left(\frac{r}{3}\right) + 432\delta_3\left(\frac{r}{6}\right) - B(r)\right]q^r - 20736\sum_{r=2i+3j}\delta(i)\delta(j)q^r = 1+\sum_{r=1}^{\infty}A(r)q^r.$$

Hence,

$$\sum_{2i+3j=r} \delta(i)\delta(j) = \left(\frac{1}{24}-\frac{r}{6}\right)\delta_1\left(\frac{r}{2}\right) + \left(\frac{1}{24}-\frac{3r}{8}\right)\delta_1\left(\frac{r}{3}\right) - \left(\frac{7}{3456}\right)\delta_3(r) + \left(\frac{65}{432}\right)\delta_3\left(\frac{r}{2}\right) + \left(\frac{37}{128}\right)\delta_3\left(\frac{r}{3}\right) - \left(\frac{1}{48}\right)\delta_3\left(\frac{r}{6}\right) + \frac{1}{20736}\left(B(r)-A(r)\right),$$

where $1 + \sum_{r=1}^{\infty} A(r)q^r = a(q^2)b^3(q^2)$ and $1 + \sum_{r=1}^{\infty} B(r)q^r = a^3(q)a(q^2)$. ∎

REFERENCES

[1] A. Alaca, S. Alaca, and K. S. Williams, "On the two-dimensional theta functions of the Borweins", Acta Arithmetica, vol. 124, no. 2, pp177-195, 2006

[2] A. Alaca, S. Alaca, K. S. Williams, "Evaluation of the convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$", Advances in Theoretical and Applied Mathematics, vol. 1, no.1, pp27-48, 2006

[3] A. Alaca, S. Alaca, and K. S. Williams, "Evaluation of the convolution sums $\sum_{l+18m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+9m=n} \sigma(l)\sigma(m)$", International Mathematical Forum. Journal for Theory and Applications, vol. 2, pp45-68, 2007

[4] S. Alaca and K. S. Williams, "Evaluation of the convolution sums $\sum_{l+6m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+3m=n} \sigma(l)\sigma(m)$ ", Journal of Number Theory, vol. 124, no. 2, pp491-510, 2007

[5] A. Alaca, S. Alaca, and K. S. Williams, "Evaluation of the convolution sums $\sum_{l+24m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+8m=n} \sigma(l)\sigma(m)$", Mathematical Journal of Okayama University, vol. 49, pp93-111, 2007

[6] A. Alaca, S. Alaca, and K. S. Williams, "The convolution sum Canadian $\sum_{m<\frac{n}{16}} \sigma(m)\sigma(n-16m)$", Mathematical Bulletin. Bulletin Canadien de Mathematiques, vol. 51, no. 1, pp3-14, 2008

[7] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer, New York, 1991

[8] B. C. Berndt, "Number Theory in the Spirit of Ramanujan", American Mathematical Society, Providence, RI, USA, vol. 34, 2006

[9] J.M. Borwein and P.B. Borwein, "A cubic counterpart of Jacobi's identity and the AGM", Transactions of the American Mathematical Society, vol. 323, no. 2, pp691-701, 1991

[10] J.M. Borwein, P.B. Browein, F.G. Garvan, "Some cubic modular identities of Ramanujan." Transactions of the American Mathematical Society, vol. 343, no. 1, pp35-47, 1994

[11] J. W. L. Glaisher, "On the square of the series in which the coeff's are the sum of the divisors of the exponents", Messenger of Mathematics, vol. 14, pp156-163, 1885

[12] Harekala Chandrashekara Vidya and Badanidiyoor Ashwath Rao, "Intriguing Relationships among Eisenstein Series, Borewein's Cubic Theta Functions, and the Class One Infinite Series", IAENG International Journal of Computer Science, vol. 50, no.4, pp1166-1173, 2023

[13] Z. G. Liu, "The Borweins' cubic theta function identity and some cubic modular identities of Ramanujan", The Ramanujan Journal, vol. 4, no. 1, pp43-50, 2000

[14] Z. G. Liu, "Some Eisenstein series identities", Journal of Number Theory, vol. 85, no. 2, pp231-252, 2000

[15] S. Ramanujan, Notebooks, Vols. 1, 2, Tata Institute of Fundamental Research, Mumbai, India, 1957

[16] Shruti and B. R. Srivatsa Kumar, "Some new Eisenstein series containing the Borweins' cubic theta functions and convolution sum $\sum_{i+4j=n} \sigma(i)\sigma(j)$", Afrika Matematika, vol. 33, pp971-982, 2020

[17] Smith, W. Steven, "The Scientist and Engineer's Guide to Digital Signal Processing". United States, California Technical Pub., 1999

[18] Smitha Ganesh Bhat, Vidya Harekala Chandrashekara, and Ashwath Rao Badanidiyoor, "Some Eisenstein Identities Involving Borweins' Cubic Theta Functions and Evaluation of Compound Convolution Sum," Engineering Letters, vol. 32, no. 3, pp452-462, 2024

[19] H. C. Vidya and B. R. Srivatsa Kumar, "Some studies on Eisenstein seies and its applications", Notes on Number Theory and Discrete Mathematics, vol. 25, no. 4, pp30-43, 2019

[20] H. C. Vidya and B Ashwath Rao, "Few more relation connecting Ramanujan type Eisenstein series and cubic theta functions of Borwein", Proceedings of the Jangjeon Mathematical Society, vol. 25, no. 2, pp253-263, 2022

[21] Vidya Harekala Chandrashekara and Smitha Ganesh Bhat, "Formulating Few Key Identities Relating Infinite Series of Eisenstein and Borweins' Cubic Theta Functions", Engineering Letters, vol. 31, no. 4, pp1383-1394, 2023

[22] K. S. Williams, "The convolution sum, $\sum_{m<\frac{n}{9}} \sigma(m)\sigma(n-9m)$", International Journal of Number Theory, vol. 1, no. 2, pp193-205, 2005

[23] K. S. Williams, "The convolution sum $\sum_{m<\frac{n}{8}} \sigma(m)\sigma(n-8m)$", Pacific Journal of Mathematics, vol. 228, no. 2, pp387-396, 2006

[24] E. X. W. Xia and O. X. M. Yao, "Eisenstein series identities involving the Borweins' cubic theta functions", Journal of Applied Mathematics, 2012, Article ID 181264.