Dynamic Behaviors of a Non-autonomous Single-Species Feedback Control System

Qin Yue, Ankur Jyoti Kashyap, and Fengde Chen

***Abstract*—China's birth rate has declined, but its mortality rate has risen year after year due to the new coronavirus pandemic. Using the new coronavirus pandemic as a feedback control variable, we proposed a new non-autonomous single-population feedback control model in which the feedback control variable reduces the population's birth rate while increasing the population's mortality rate. We determined sufficient conditions for the persistence, extinction, and global stability. The analytical results are then compared numerically with relevant examples.**

***Index Terms*—species, feedback control, global stability, new coronavirus pandemic**

I. Introduction

FEEDBACK control ecosystems, where feedback mechanisms play a crucial role in regulating the dynamics and stability of the ecosystem, have received significant attention from researchers and scientists in recent years ([1]-[43]).

Gopalsamy and Weng [23] investigated the following single-species feedback control ecosystem considering logistic growth with a delay.

$$\begin{aligned} \dot{n} &= rn\Big[1-\frac{a_1n(t)+a_2n(t-\tau)}{K}-cu(t)\Big], \\ \dot{u} &= -au(t)+bn(t). \end{aligned} \tag{1}$$

They determined the sufficient conditions for the global attractivity of the interior equilibrium of (1).

Fan et al. [32] analyzed a non-autonomous, time-delayed periodic feedback control system and established an existence criterion for the interior equilibrium. Later, Chen et al. [24] explored the persistence of the general non-autonomous case in [32]. In the context of stage structure, Y. Z. Yang [22] proposed a single population feedback control model and explored the existence of positive periodic solutions. Chen et al. [25] revisited the persistence of the positive equilibrium for the general non-autonomous case in [22]. Their study revealed that the feedback control variable has no impact on persistence, but it negatively correlates with population density. Using numerical simulations, they demonstrated that the density of the population decreases as the feedback control variable increases, which increases the likelihood of population extinction. Using human stocking as a feedback control variable, Yue [35] proposed for the first time a single-population model with positive feedback control and observed that a limited stocking may not help to control the population's extinction. However, a sufficiently large stocking could help the population continue to survive. This highlights the potential efficacy of positive feedback control as a crucial approach to prevent species extinction. In the context of discrete systems, Li and Zhu[33] investigated the positive periodic solution of a discrete system with feedback control for a single species. Later, Chen[34] presented the sufficient conditions for the permanence of the model. Across various terrestrial and aquatic species, a positive association exists between population size and individual fitness, commonly referred to as the Allee effect. Any species may exhibit the Allee effect in particular circumstances because low population density can make it difficult for endangered species to locate mates. Considering the fact Zhu et al.[2] proposed a single-species system combining feedback control and the Allee effect and demonstrated Bogdanov-Takens and saddle-node bifurcations.

Human capture or release is considered the feedback control variable in all of the feedback control models discussed above. But in reality, it is not always necessarily the capture rate to be the control variable. Fear of predation risk is one of the significant factors associated with the predation process. The prey population dynamics are more affected by indirect interaction with a predator than by direct killing. Prey often compromises with their food source and foraging strategies, affecting their community growth by lowering the fertility rate in the long run. Experimental results on the impact of predation risk on prey reproduction were reported by Zanette et al. [44], Elliott et al. [45]. Considering the impact of fear on prey reproduction, several exciting works have been proposed in recent years [47], [48], [49], [50]. However, in a recent experiment on snowshoe hares, MacLeod et al. [46] revealed that the fear induced in the prey due to predation risk could also be lethal in wild animals. Such lethal scenarios of fear have also been observed recently in the human community. The new crown pneumonia, also known as COVID-19, has had a significant negative impact on the daily lives of individuals in the last few years, particularly in China. During 2022, the annual birth population was 9.56 million, whereas the death population reached 10.41 million, resulting in a decrease of 85 million from the year-end. According to the National Bureau of Statistics of China, at the end of the year 2023, the population of China was 1,409,670,000 individuals, which is 2,080,000 individuals less than that at the end of the previous year. The annual births amounted to 9.02 million, yielding a birth rate of 6.39 per thousand. Meanwhile, there were 11.1 million deaths, resulting in a mortality rate of 7.87 per thousand. Consequently, the natural population growth rate stood at -1.48 per thousand. In response to population decline, certain cities in China initiated campaigns to promote multiple births. Several factors contributed to the decline in the birth population, with one factor being

Manuscript received May 3, 2023; revised April 27, 2024. This work is supported by the Provincial quality engineering project(2022cxtd167).

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associated with the COVID-19 pandemic. The pandemic has heightened feelings of insecurity and uncertainty about the future, leading families to hesitate about future fertility and diminishing people's desire to have children. The COVID-19 pandemic has directly or indirectly lowered the income of family members, exacerbating the impact of childcare costs on fertility. During the outbreak, there were many enterprises, tiny and micro-enterprise units, which not only affected the income of family members but also increased the conflicts between work and family responsibilities, increasing childbirth panic or fear. Of course, some people worry that these routine services, such as maternal care, maternal inspections, and so on, will be affected during the epidemic. These factors prompted many individuals to choose to postpone or even cancel birth plans. Moreover, the mortality rate was somewhat increased by the absence of a timely response program and the strain on medical resources during the initial stages of the new coronavirus outbreak. The combination of underlying severe diseases, such as malignant tumours, kidney failure, etc., being infected with the new coronavirus can worsen their condition, resulting in higher mortality rates. Recently, Yue and Chen [38] proposed a new single-population feedback control model using the new crown epidemic as the feedback control variable as follows:

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a}{1+k_1u} - b(1+k_2u) - cx\Big),\\ \frac{du}{dt} &= -eu + fx.\end{aligned} \tag{2}$$

where u represents the feedback control variable. In the system (2), the birth and death rates of the species were assumed by incorporating the fear effect. The birth and death rates were presented by the terms $\frac{a}{1+k_1u}$ and $b(1+k_2u)$, where k_1, k_2 represents the strengths of fear. Such assumptions came from recent works on predator-prey systems with fear effects on prey species[28]-[31]. System (2) possesses only one globally asymptotically stable positive equilibrium, unaffected by the feedback control variable in a species where the reproduction rate is greater than the extinction rate. However, the ultimate density of the species decreases with increasing feedback control. In the system (2), for $a > b$, the system admits a unique positive equilibrium that is globally asymptotically stable, while for the case $a < b$, the system will be driven to extinction.

It is well known that the human environment is constantly changing over time. Hence a suitable system needs to assume that the coefficients of the system are time-varying, which inspires us to explore the following non-autonomous system:

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a(t)}{1+k_1(t)u} - b(t)(1+k_2(t)u)\\ &\qquad -c(t)x\Big),\\ \frac{du}{dt} &= -e(t)u + f(t)x.\end{aligned} \tag{3}$$

Now, for a continuous and bounded function, we let $f^l = \inf_{t\in R} f(t)$ and $f^u = \sup_{t\in R} f(t)$.

The system (3) is biologically feasible under the following assumption:

(H_1) $a(t), k_1(t), k_2(t), c(t), e(t)$ and $f(t)$ are all continuous and strictly positive functions that meet

$$\min\{a^l, k_1^l, k_2^l, c^l, e^l, f^l\} > 0,$$

$$\max\{a^u, k_1^u, k_2^u, c^u, e^u, f^u\} < +\infty.$$

The objective of this study is to examine the dynamical behaviours of the system (3). Including time-dependent parameters distinguishes our research from previous studies, thus introducing a novel aspect to our model.

This paper is arranged as follows.
The persistence of the proposed model is analyzed in the following section. Section III deals with the results related to the extinction of the system. In Section IV, the global attractivity of the system is explored by constructing an appropriate Lyapunov function. Numerical simulations are conducted in Section V to demonstrate the viability of the primary findings. Finally, the key findings of this study are summarized in Section VI.

II. Permanence

In feedback control models, persistence indicates that the species' population remains within a viable range, neither growing excessively nor declining to extinction. This helps predict the species' resilience to environmental changes and design effective management strategies for conservation and population control. The following theorem is an outcome of the persistence of system (3).

Theorem 2.1. *Assumes that*

$$\frac{a^l}{1+k_1^uM_2} > b^u(1+k_2^uM_2) \tag{4}$$

holds, where

$$M_2 = \frac{f^u\Big(\frac{a^u-b^l}{c^l}\Big)}{e^l},$$

then system (3) is permanent.

Proof. It is implied by condition (4) that the following inequality holds for sufficiently small $\varepsilon > 0$:

$$\frac{a^l}{1+k_1^u(M_2+\varepsilon)} > b^u(1+k_2^u(M_2+\varepsilon)) \tag{5}$$

Using the system (3)'s first equation, one has

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a(t)}{1+k_1(t)u} - b(1+k_2(t)u) - c(t)x\Big)\\ &\le x\Big(a(t) - b(t) - c(t)x\Big)\\ &\le x\Big(a^u - b^l - c^lx\Big).\end{aligned} \tag{6}$$

Consequently,

$$\limsup_{t\to+\infty} x(t) \le \frac{a^u-b^l}{c^l} \stackrel{def}{=} M_1. \tag{7}$$

For $\varepsilon > 0$ which satisfies inequality (5), from (7) there exists $T_1 > 0$ such that

$$x(t) < \frac{a^u-b^l}{c^l} + \varepsilon \;\; for \;\; all \;\; t \ge T_1. \tag{8}$$

For $t > T_1$, one has the following from the second equation of (3) and (8)

$$\begin{aligned}\frac{du}{dt} &= -e(t)u + f(t)x \\ &\le -e^l u + f^u\Big(\frac{a^u - b^l}{c^l} + \varepsilon\Big).\end{aligned} \tag{9}$$

Thus,

$$\limsup_{t\to+\infty} u(t) \le \frac{f^u\Big(\frac{a^u - b^l}{c^l} + \varepsilon\Big)}{e^l} \overset{def}{=} M_2. \tag{10}$$

Since ε is arbitrary small positive constant, setting $\varepsilon \to 0$ in (10) results in

$$\limsup_{t\to+\infty} u(t) \le \frac{f^u\Big(\frac{a^u - b^l}{c^l}\Big)}{e^l}. \tag{11}$$

For $\varepsilon > 0$ which satisfies inequality (5), there exists $T_2 > T_1$ from (11) such that

$$u(t) < \frac{f^u\Big(\frac{a^u - b^l}{c^l}\Big)}{e^l} + \varepsilon \overset{def}{=} M_2 + \varepsilon \ \ for\ \ all\ \ t \ge T_2. \tag{12}$$

From the first equation of the system (3), one has

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a(t)}{1 + k_1(t)u} - b(1 + k_2(t)u) - c(t)x\Big) \\ &\ge x\Big(\frac{a^l}{1 + k_1^u u} - b^u(1 + k_2^u u) - c^u x\Big) \\ &\ge x\Big(\frac{a^l}{1 + k_1^u(M_2 + \varepsilon)} - c^u x \\ &\quad -b^u(1 + k_2^u(M_2 + \varepsilon))\Big).\end{aligned} \tag{13}$$

Consequently,

$$\liminf_{t\to+\infty} x(t) \ge \frac{\Delta_\varepsilon}{c^u}, \tag{14}$$

where

$$\Delta_\varepsilon = \frac{a^l}{1 + k_1^u(M_2 + \varepsilon)} - b^u(1 + k_2^u(M_2 + \varepsilon)).$$

Given an arbitrary small positive constant $\varepsilon > 0$, taking the limit as ε approaches 0 in equation (15) results in:

$$\liminf_{t\to+\infty} x(t) \ge \frac{\Delta}{c^u} \overset{def}{=} m_1. \tag{15}$$

where

$$\Delta = \frac{a^l}{1 + k_1^u M_2} - b^u(1 + k_2^u M_2).$$

For $\varepsilon_1 > 0$ small enough, without loss of generality, we may assume that $\varepsilon_1 < \frac{1}{2}\frac{\Delta}{c^u}$, from (15) there exits a $T_3 > T_2$ such that

$$x(t) > \frac{\Delta}{c^u} - \varepsilon_1 \ \ for\ \ all\ \ t \ge T_3. \tag{16}$$

Based on the second equation of equation (1.9) and equation (2.12), the following relationship holds for time greater than T_3,

$$\begin{aligned}\frac{du}{dt} &= -e(t)u + f(t)x. \\ &\ge -e^u u + f^l\Big(\frac{\Delta}{c^u} - \varepsilon_1\Big).\end{aligned} \tag{17}$$

Hence,

$$\liminf_{t\to+\infty} u(t) \ge \frac{f^l\Big(\frac{\Delta}{c^u} - \varepsilon_1\Big)}{e^u}. \tag{18}$$

Given that ε_1 is a positive constant with infinitesimal value, taking the limit as ε_1 approaches zero in equation (18) results in

$$\liminf_{t\to+\infty} u(t) \ge \frac{f^l\Delta}{c^u e^u} \overset{def}{=} m_2. \tag{19}$$

Equations (7), (11), (15), and (19) provide evidence that, given the premise that equation (4) is valid, the system exhibits permanence. The proof of Theorem 2.1 is concluded at this point.

III. Extinction

This section aims to investigate the parametric conditions associated with the extinction of the system. More precisely, we prove the following theorem.

Theorem 3.1. *Assumes that*

$$a^u < b^l, \tag{20}$$

then system (3) extincts, i.e.,

$$\lim_{t\to+\infty} x(t) = 0, \ \ \lim_{t\to+\infty} u(t) = 0. \tag{21}$$

Proof. By considering the first equation in the system (3), it can be observed that

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a(t)}{1 + k_1(t)u} - b(1 + k_2(t)u) - c(t)x\Big) \\ &\le x\Big(a(t) - b(t) - c(t)x\Big) \\ &\le x\Big(a^u - b^l\Big).\end{aligned} \tag{22}$$

Hence, it follows from (20) that

$$x(t) \le x(0)\exp\{(a^u - b^l)t\} \to 0 \ \text{ as } \ t \to +\infty. \tag{23}$$

For $\varepsilon > 0$ enough small, from (23) there exists a $T_5 > 0$ such that

$$x(t) < \varepsilon \ \ for\ \ all\ \ t \ge T_5. \tag{24}$$

Based on the second equation of equation (3) and equation (24), it may be concluded that for $t > T_5$.

$$\begin{aligned}\frac{du}{dt} &= -e(t)u + f(t)x. \\ &\le -e^l u + f^u\varepsilon,\end{aligned} \tag{25}$$

and so,

$$\limsup_{t\to+\infty} u(t) \le \frac{f^u\varepsilon}{e^l}. \tag{26}$$

Since ε is arbitrary small positive constant, setting $\varepsilon \to 0$ in (26) leads to

$$\limsup_{t\to+\infty} u(t) \le 0. \tag{27}$$

On the other hand, based on the positivity of the solution, it can be inferred that

$$\liminf_{t\to+\infty} u(t) \ge 0. \tag{28}$$

(27) combine with (28) leads to

$$\lim_{t\to+\infty} u(t) = 0. \tag{29}$$

Equations (23) and (29) demonstrate that, given the premise that equation (20) is valid, the system will become extinct. The proof of Theorem 3.1 is concluded at this point.

IV. Global attractivity

In this section, we will explore the problem of global attractiveness of the positive solutions of the system, and in fact, we obtained the following result.

Theorem 4.1 *Let $(x^*(t), u^*(t))$ be a bounded positive solution of system (3). If*

$$\frac{a^l}{1+k_1^u M_2} > b^u(1+k_2^u M_2), \tag{30}$$

$$e^l > \frac{a^u k_1^u}{(1+k_1^l m_2)^2} + b^u k_2^u \tag{31}$$

and

$$c^l > f^u \tag{32}$$

hold, the variables m_2 and M_2 are specified by equations (10) and (19) correspondingly. Then $(x^*(t), u^*(t))$ exhibits global asymptotic stability.

Proof. The condition expressed in inequality (31) indicates that, for a sufficiently small positive constant $\varepsilon > 0$, it is possible to assume, without loss of generality, that $\varepsilon < \frac{1}{2}\min\{m_2, m_1\}$. Under this assumption, the following inequality is valid.

$$e^l > \frac{a^u k_1^u}{(1+k_1^l(m_2-\varepsilon))^2} + b^u k_2^u. \tag{33}$$

Consider the solution $(x(t), u(t))^T$ of equation (3) with a positive initial value. It may be deduced from condition (30) and Theorem 2.1 that, given any positive value of ε, there exists a positive value of T such that for all values of t greater than or equal to T,

$$\begin{array}{c} m_1 - \varepsilon < x(t), x^*(t) < M_1 + \varepsilon, \\ m_2 - \varepsilon < x(t), x^*(t) < M_2 + \varepsilon. \end{array} \tag{34}$$

Let us consider a Lyapunov function that is defined by

$$V(t) = |\ln\{x(t)\} - \ln\{x^*(t)\}| + |u(t) - u^*(t)|, t \geq 0. \tag{35}$$

We are now estimating and computing the upper right derivative of $V(t)$ along the system (3) solutions for $t > T$. Applying (34) yields the following results:

$$\begin{aligned} & D^+V(t) \\ = \; & sgn(x(t)-x^*(t))\Big[-\frac{a(t)}{1+k_1(t)u^*(t)} \\ & +b(t)(1+k_2(t)u^*(t)) - b(t)(1+k_2(t)u(t)) \\ & +\frac{a(t)}{1+k_1(t)u(t)} + c(t)x^*(t) - c(t)x(t)\Big] \\ & +sgn(u(t)-u^*(t))\Big[e(t)u^*(t) - f(t)x^*(t) \\ & -e(t)u(t) + f(t)x(t)\Big] \\ = \; & sgn(x(t)-x^*(t))\Big[\frac{a(t)k_1(t)(u(t)-u^*(t))}{(1+k_1(t)u^*(t))(1+k_1(t)u(t))} \\ & -b(t)k_2(t)(u(t)-u^*(t)) - c(t)(x(t)-x^*(t))\Big] \\ & +sgn(u(t)-u^*(t))\Big[-e(t)(u(t)-u^*(t)) \\ & +f(t)(x(t)-x^*(t))\Big] \\ \leq \; & -A_1|x(t)-x^*(t)| - A_2|u(t)-u^*(t)|, \end{aligned} \tag{36}$$

where

$$\begin{aligned} A_1 &= c^l - f^u > 0, \\ A_2 &= e^l - \frac{a^u k_1^u}{(1+k_1^l(m_2-\varepsilon))^2} - b^u k_2^u > 0. \end{aligned} \tag{37}$$

For $t \geq T$, one thus has

$$D^+V(t) \leq -\mu\Big(|x(t)-x^*(t)| + |u(t)-u^*(t)|\Big), \tag{38}$$

where $\mu = \min\{A_1, A_2\}$. Performing integration on both sides of equation (38) with respect to the variable t across the interval from T to t yields

$$\begin{aligned} V(t) \quad & +\mu\int_T^t \Big(|x(s)-x^*(s)| + |u(s)-u^*(s)|\Big)ds \\ & \leq V(T) < +\infty, \;\; t \geq T. \end{aligned}$$

Then, for all $t \geq T$,

$$\int_T^t \Big(|x(s)-x^*(s)| + |u(s)-u^*(s)|\Big)ds \leq \mu^{-1}V(T) < +\infty,$$

and hence,

$$|x(t)-x^*(t)| + |u(t)-u^*(t)| \in L^1([T, +\infty)).$$

The fact that $x^*(t)$ and $u^*(t)$ are bounded, and that $x(t)$ and $u(t)$ are ultimately bounded, implies that the derivatives of $x(t)$, $x^*(t)$, $u(t)$, and $u^*(t)$ are all bounded for $t \geq T$, as indicated by the equations that govern their behavior. Consequently, it may be inferred that $|x(t)-x^*(t)|+|u(t)-u^*(t)|$ is uniformly continuous on $[T, +\infty)$. Thus, by Barbălat's Lemma[37], we have

$$\lim_{t\to+\infty}\Big(|x(t)-x^*(t)| + |u(t)-u^*(t)|\Big) = 0.$$

The proof is completed.

V. Numeric simulations

In this section, we conducted numerical simulations to compare the analytical findings of the system (3). Let us examine the subsequent three illustrations.

Example 5.1

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{1+\frac{1}{2}\cos(t)}{1+(2+\sin(t))u} \\ &\quad -2(1+(1+0.3\cos(t))u)-2x\Big)x, \qquad (39) \\ \frac{du}{dt} &= -u+x.\end{aligned}$$

In accordance with the system (3), we adopt $a(t) = 1+\frac{1}{2}\cos(t), b(t) = 2, k_1(t) = 2+\sin(t), k_2(t) = 1+0.3\cos(t), c = 2, e = f = 1$, then,

$$a^u = \frac{3}{2} < 2 = b^l,$$

hence, it follows from Theorem 4.1 that the system is extinct. Fig. 1 and 2 support this assertion.

Example 5.2

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{2}{1+(0.2+0.1\sin(t))u} \\ &\quad -(1+(0.2+0.1\cos(t))u)-x\Big), \qquad (40) \\ \frac{du}{dt} &= -2u+\tfrac{1}{2}x,\end{aligned}$$

In accordance with the system (3), we adopt $a(t) = 2, b(t) = 1, k_1(t) = 0.2+0.1\sin(t), k_2(t) = 0.2+0.1\cos(t), c = 1, e = 2, f = \frac{1}{2}$, then,

$$\frac{a^l}{1+k_1^u M_2} \approx 1.86 > 1.075 = b^u(1+k_2^u M_2), \qquad (41)$$

$$e^l = 2 > 0.277 \approx \frac{a^u k_1^u}{(1+k_1^l m_2)^2} + b^u k_2^u, \qquad (42)$$

and

$$c^l = 1 > \frac{1}{2} = f^u \qquad (43)$$

All the conditions of Theorems 2.1 and 4.1 are satisfied, so the positive solution of system (40) is globally asymptotically stable. Fig 3 and 4 support this assertion.

Example 5.3

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{2}{1+(3+\sin(t))u} \\ &\quad -(1+(3+\cos(t))u)-x\Big), \qquad (44) \\ \frac{du}{dt} &= -u+2x,\end{aligned}$$

In accordance with the system (3), we adopt $a(t) = 2, b(t) = 1, k_1(t) = 3+\sin(t), k_2(t) = 3+\cos(t), c = 1, e = 1, f = 2$, then,

$$c^l = 1 < 2 = f^u \qquad (45)$$

That is, if condition (32) in Theorem 4.1 does not hold, we have no idea what the stability property of this system is, however, numeric simulations (Fig 5 and 6) show that in this case, the positive solution of system (44) is globally asymptotically stable.

Example 5.4

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{3}{1+u}-2(1+u)-x\Big), \\ \frac{du}{dt} &= -(\tfrac{3}{2}+\tfrac{1}{2}\cos(t))u+(\tfrac{3}{2}-\tfrac{1}{2}\sin(t))x,\end{aligned} \qquad (46)$$

In accordance with the system (3), we adopt $a(t) = 3, b(t) = 2, k_1(t) = 1, k_2(t) = 1, c = 1, e = \frac{3}{2}-\frac{1}{2}\sin(t), f = \frac{3}{2}+\frac{1}{2}\cos(t)$, then, by simple computation, we have $M_2 = 2$, and

$$\frac{a^l}{1+k_1^u M_2} = 1 < 6 = b^u(1+k_2^u M_2). \qquad (47)$$

In this case, condition (4) in Theorem 2.1 is not satisfied, we have no idea what the persistent property of this system is. However, numeric simulations (Fig 7) show that in this case, system (47) is permanent.

VI. Discussion

In recent years, China's newborn population has dropped sharply. In contrast, population deaths have increased yearly, a large part of which is caused by the epidemic, which inspired us to use the coronavirus as a feedback control variable and put forward a system (3). We explored dynamical behaviours such as system persistence, extinction, and global attractiveness of the proposed system analytically and numerically. We obtained sufficient conditions for the global stability and persistence of the proposed system. Figures 5 and 6 demonstrate although the conditions for global stability are not satisfied, the system may exhibit globally stable dynamical behaviour, i.e., it may have a large basin of attraction. Numerical results also revealed that although the persistence conditions are not satisfied, the system may be persistent (Figure 7). These two numerical results indicate that our results (Theorem 2.1 and 4.1) still have scope for improvement. However, with the current research method, it isn't easy to obtain more in-depth results. In the present study, the stability of non-autonomous systems is mainly studied by constructing Lyapunov functions, which makes it impossible to produce results that are too precise. Suppose we do not explore the stability of the system and instead study the more realistic topic, such as persistence property. In that case, we may be able to come up with the following conjecture: condition $a^l > b^u$ is enough to ensure the persistence of the system. We will explore this issue further in a subsequent paper.

References

[1] Y. Lv, L. Chen, F. Chen, "Stability and bifurcation in a single species logistic model with additive Allee effect and feedback control," Advances in Difference Equations, vol. 2020, article number: 129, 2020.

[2] Z. Zhu, M. He, Z. Li, et al, "Stability and bifurcation in a Logistic model with Allee effect and feedback control," International Journal of Bifurcation and Chaos, vol. 30, no. 15, 2050231, 2020.

[3] F. Chen, Y. Chong, S. Lin, "Global stability of a commensal symbiosis model with Holling II functional response and feedback controls," Wseas Trans. Syst. Contr, vol. 17, no.1, pp. 279-286, 2022.

[4] K. Yang, Z. Miao, F. Chen, et al., "Influence of single feedback control variable on an autonomous Holling-II type cooperative system," Journal of Mathematical Analysis and Applications, vol. 435, no. 1, pp.874-888, 2016.

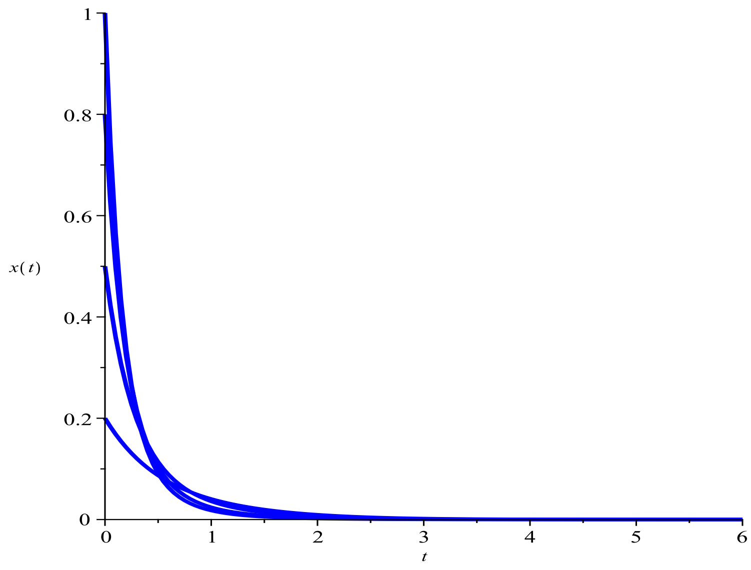

Fig. 1. Numeric simulations of $x(t)$ in the system (39), the initial condition $(x(0), u(0)) = (1, 1)$, $(0.8, 0.8)$, $(0.2, 0.2)$ and $(0.5, 0.5)$, respectively.

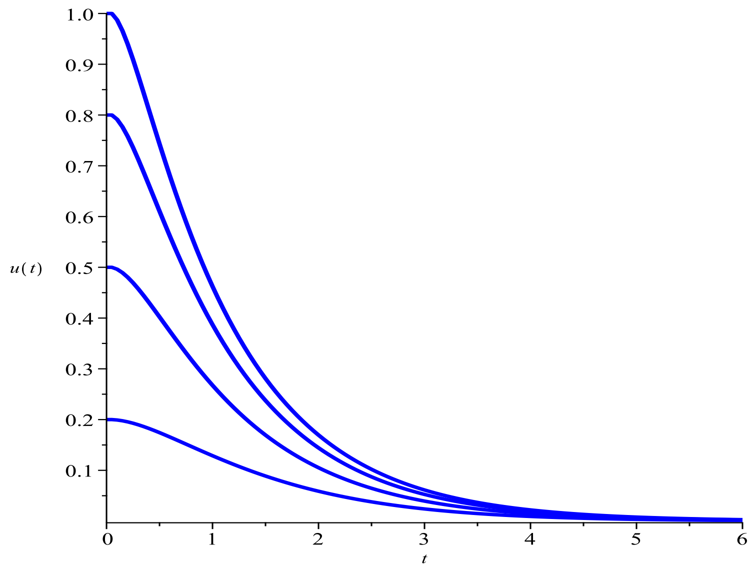

Fig. 2. Numeric simulations of $u(t)$ in the system (39), the initial condition $(x(0), u(0)) = (1, 1)$, $(0.8, 0.8)$, $(0.2, 0.2)$ and $(0.5, 0.5)$, respectively.

[5] R. Han, F. Chen, "Global stability of a commensal symbiosis model with feedback controls," Commun. Math. Biol. Neurosci., vol. 2015, article ID 15, 2015.
[6] Z. Li, M. Han, F. Chen, "Influence of feedback controls on an autonomous Lotka–Volterra competitive system with infinite delays," Nonlinear Analysis: Real World Applications, vol. 14, no. 1, pp. 402-413, 2013.
[7] Z. Li, M. Han, et al., "Global stability of a predator-prey system with stage structure and mutual interference," Discrete and Continuous Dynamical Systems-Series B, vol. 19, no. 1, pp.173-187, 2014.
[8] L. Chen, F. Chen, "Extinction in a discrete Lotka–Volterra competitive system with the effect of toxic substances and feedback controls," International Journal of Biomathematics, vol. 8, no.01, 1550012, 2016.
[9] R. Han, F. Chen, X. Xie, et al., "Global stability of May cooperative system with feedback controls," Advances in Difference Equations, vol. 2015, article number:360, 2015.
[10] R. Y. Han, X. D. Xie, F. D. Chen, "Permanence and global attractivity of a discrete pollination mutualism in plant-pollinator system with feedback controls," Advances in Difference Equations, vol. 2016, article number:199, 2016.
[11] J. Xu, F. Chen, "Permanence of a Lotka-Volterra cooperative system with time delays and feedback controls," Commun. Math. Biol. Neurosci., vol. 2015, article ID 18, 2015.
[12] Y. Xue, X. Xie, Q. Lin, et al., "Global attractivity and extinction of a discrete competitive system with infinite delays and single feedback control," Discrete Dynamics in Nature and Society, vol. 2018, article ID 1893181, 2018.
[13] L. Zhao, X. Xie, L. Yang, et al., "Dynamic behaviors of a discrete Lotka-Volterra competition system with infinite delays and single feedback control," Abstract and Applied Analysis, vol. 2014, article ID 867313, 2014.
[14] Z. Miao, F. Chen, J. Liu, et al., "Dynamic behaviors of a discrete Lotka-Volterra competitive system with the effect of toxic substances and feedback controls," Advances in Difference Equations, vol. 2017, article number: 112, 2017.
[15] Y. Wang, "Periodic and almost periodic solutions of a nonlinear single species discrete model with feedback control," Applied Mathematics and Computation, vol. 219, no.10, pp. 5480-5486, 2013.
[16] Y. Li, L. Yang, H. Zhang, "Permanence and uniformly asymptotical stability of almost periodic solutions for a single-species model with

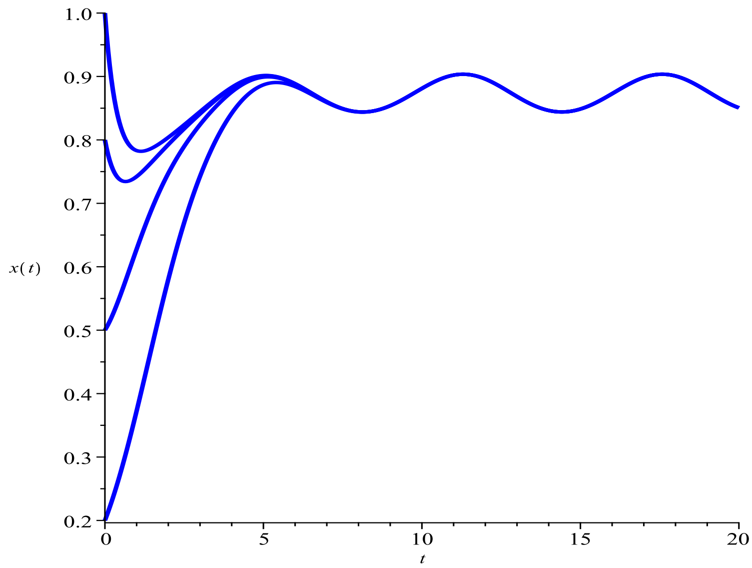

Fig. 3. Numeric simulations of $x(t)$ in the system (40), the initial condition $(x(0), u(0)) = (1, 1)$, $(0.8, 0.8)$, $(0.2, 0.2)$ and $(0.5, 0.5)$, respectively.

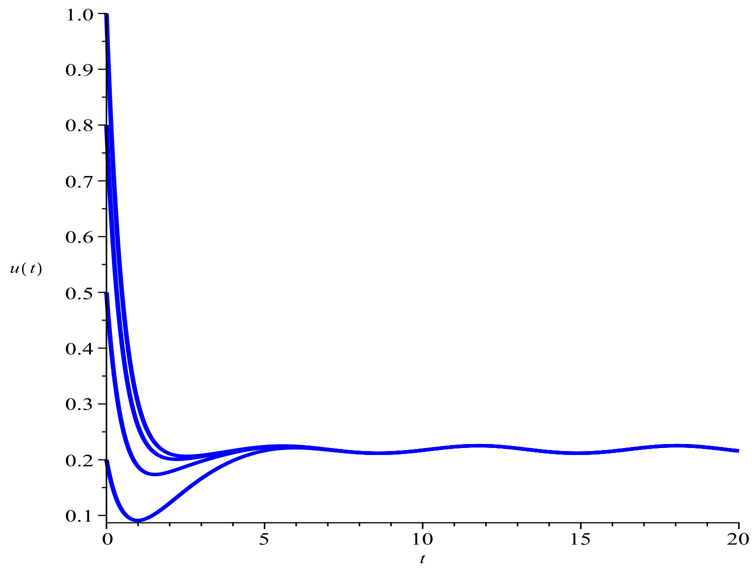

Fig. 4. Numeric simulations of $u(t)$ in the system (40), the initial condition $(x(0), u(0)) = (1, 1)$, $(0.8, 0.8)$, $(0.2, 0.2)$ and $(0.5, 0.5)$, respectively.

feedback control on time scales," Asian-European Journal of Mathematics, vol. 07, no. 1, 1450004, 2014.

[17] Y. Li, X. Han, "Almost periodic solution for a N-species competition model with feedback controls on time scales," Journal of Applied Mathematics & Informatics, vol. 31, no.1-2, pp. 247-262, 2013.

[18] Q. Lin, "Stability analysis of a single species logistic model with Allee effect and feedback control," Advances in Difference Equations, vol. 2018, article number: 190, 2018.

[19] F. D. Chen, H. Lin, Q. Zhu, Q. Q. Li, "The permanence of a nonautonomous single-species model with stage-structure and feedback control," WSEAS Transactions on Circuits and Systems, vol. 21, pp. 258-267, 2022

[20] M. Khuddush, K. Rajendra Prasad, "Permanence and stability of multi-species nonautonomous Lotka–Volterra competitive systems with delays and feedback controls on time scales," Khayyam Journal of Mathematics, vol. 7, no. 2, pp. 241-256, 2021.

[21] Z. Jiang, A. Muhammadhaji, C. Hu, et al., "Dynamics of n-Species cooperation models with feedback controls and continuous delays," Qualitative Theory of Dynamical Systems, vol.22, no.1, 41, 2023.

[22] Y. Z. Yang, "Positive periodic solution of a nonautonomous single-species model with stage structure and feedback control," Journal of Shenyang University (Natural Science), vol. 30, no.6, pp. 512-515, 2018.

[23] K. Gopalsamy, P. X. Weng, "Feedback regulation of Logistic growth," Inthernational Journal of Mathematics Sciencess, vol.16, no. 1, pp.177-192, 1993.

[24] F. Chen, J. Yang, L. Chen, "Note on the persistent property of a feedback control system with delays," Nonlinear Analysis: Real World Applications, vol. 11, no. 2, pp. 1061-1066, 2010.

[25] F. Chen, H. Lin, Q. Zhu, Q. Q. Li, "The permanence of a nonautonomous single-species model with stage-structure and feedback control," WSEAS Transactions on Circuits and Systems, vol. 21, no.1, pp. 258-267, 2022.

[26] Y. C. Zhou, Z. Jin, J. L. Qin, Ordinary Differential Equation and Its Application, Beijing, Science Press, 2003.

[27] L. Lai, Z. Zhu, F. Chen, "Stability and bifurcation in a predator–prey model with the additive Allee effect and the fear effect," Mathematics, vol.8, no.8, 1280, 2020.

[28] T. Liu, L. Chen, F. Chen, et al., "Stability analysis of a Leslie–Gower model with strong Allee effect on prey and fear effect on predator,"

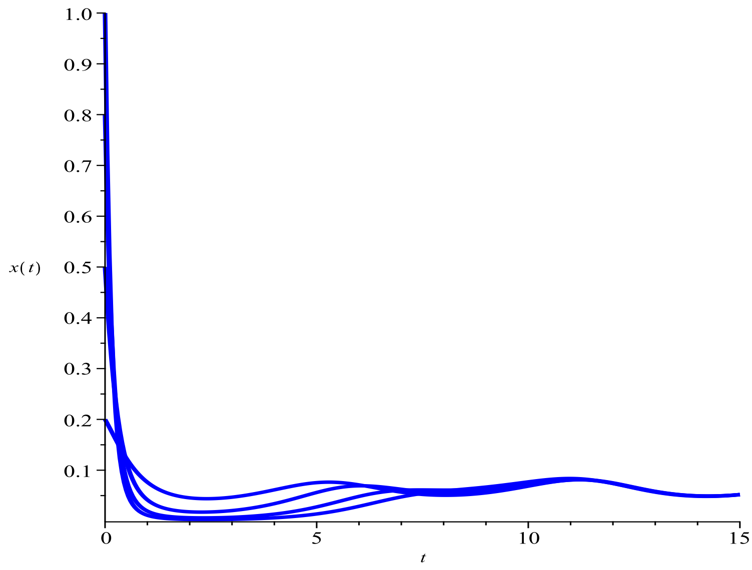

Fig. 5. Numeric simulations of $x(t)$ in the system (44), the initial condition $(x(0), u(0)) = (1, 1)$, $(0.8, 0.8)$, $(0.2, 0.2)$ and $(0.5, 0.5)$, respectively.

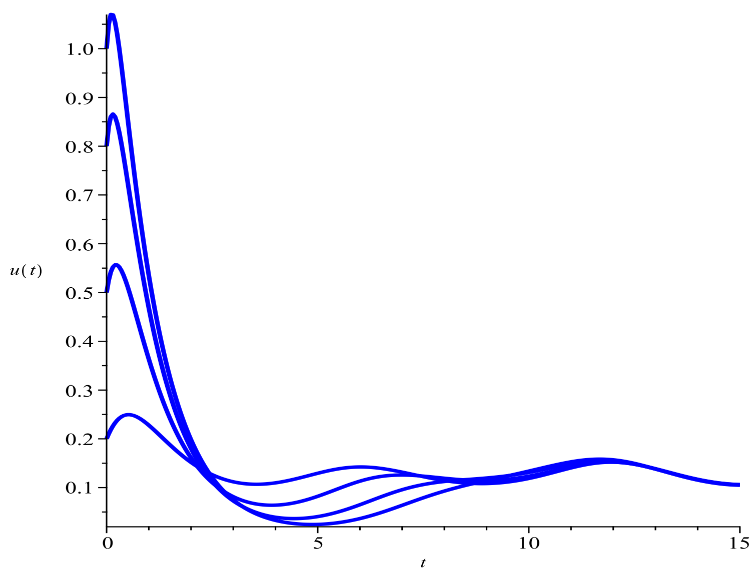

Fig. 6. Numeric simulations of $u(t)$ in the system (44), the initial condition $(x(0), u(0)) = (1, 1)$, $(0.8, 0.8)$, $(0.2, 0.2)$ and $(0.5, 0.5)$, respectively.

International Journal of Bifurcation and Chaos, vol. 32, no.6, 2250082, 2022.

[29] J. Chen, X. He, F. Chen, "The influence of fear effect to a discrete-time predator-prey system with predator has other food resource," Mathematics, vol. 9, no. 8, 865, 2021.

[30] S. Lin, F. Chen, Z. Li, et al., "Complex dynamic behaviors of a modified discrete Leslie–Gower predator–prey system with fear effect on prey species," Axioms, vol. 11, no. 10, 520, 2022.

[31] F. Chen, S. Lin, S. Chen, et al., "A new consideration of the influence of shelter on the kinetic behavior of the Leslie-Gower predator prey system with fear effect," WSEAS Transactions on Systems, vol. 22, no.1, pp. 7-18, 2023.

[32] G. H. Fan, Y. K. Li, M. C. Qin, "The existence of positive periodic solutions for periodic feedback control systems with delays," Zeitschrift für Angewandte Mathematik und Mechanik, vol.84, no.6, pp. 425–430, 2004.

[33] Y. K. Li, L. F. Zhu, "Existence of positive periodic solutions for difference equations with feedback control," Applied Mathematics Letters, vol.18, no.1, pp. 61–67, 2005.

[34] F. Chen, "Permanence of a single species discrete model with feedback control and delay," Applied Mathematics Letters, vol. 28, no.7, pp. 729-733, 2007

[35] Q. Yue, "The influence of positive feedback control to a single species stage structure system," Engineering Letters, vol. 28, no.2, pp. 322-330, 2020.

[36] F. Chen, Z. Li, Y. Huang, "Note on the permanence of a competitive system with infinite delay and feedback controls," Nonlinear Analysis: Real World Applications, vol. 8, no.2, pp. 680-687, 2007.

[37] I. Barbălat, Systems d'equations differential d'oscillations nonlinearies, Rev. Roumaine Math. Pure Appl., vol. 4, no. 2, pp. 267-270, 1959.

[38] Q. Yue, F. Chen, "A new formulation of the single species feedback control system," International Conference on Control, Artificial Intelligence, Robotics & Optimization (ICCAIRO). IEEE, pp. 118-123, 2023.

[39] C. Huang, F. Chen, Q. Zhu, et al., "How the wind changes the Leslie-Gower predator-prey system? " IAENG International Journal of Applied Mathematics, vol.53, no.3, pp. 907-915, 2023.

[40] Z. Zhu, F. Chen, L. Lai, et al., "Dynamic behaviors of a discrete May type cooperative system incorporating Michaelis-Menten type harvesting, " IAENG International Journal of Applied Mathematics, vol.50, no.3, pp. 458-467, 2020.

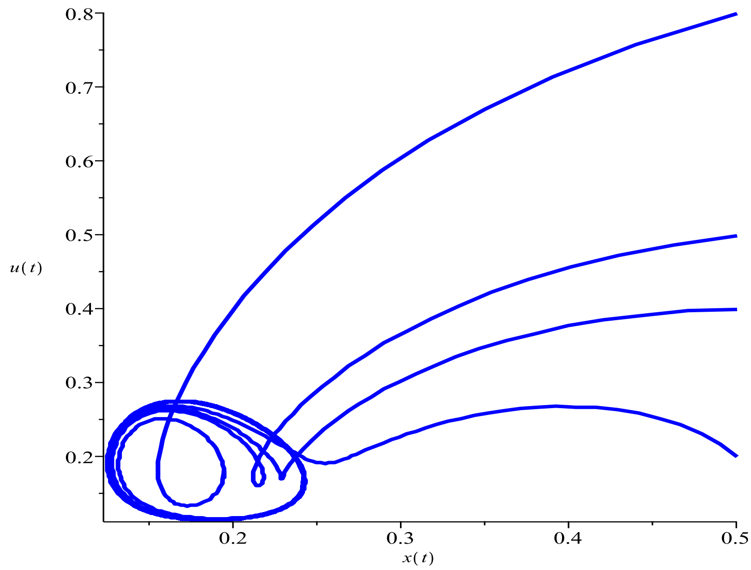

Fig. 7. Phase portraits of $(x(t), u(t))$ in the system (45), the initial condition $(x(0), u(0)) = (0.5, 0.4)$, $(0.5, 0.8)$, $(0.5, 0.2)$ and $(0.5, 0.5)$, respectively.

[41] K. Fang, J. Chen, Z. Zhu Z, et al., "Qualitative and bifurcation analysis of a single species Logistic model with Allee effect and feedback control," IAENG International Journal of Applied Mathematics, vol.52, no.2, pp. 320-326, 2022.
[42] F. D. Chen, Y. B. Chong, S. M. Chen, "Permanence and global attractivity of a non-autonomous single species system with Michaelis-Menten-type feedback control," WSEAS Transactions on Systems, vol. 22, pp. 512-519, 2023.
[43] F. Chen, Y. Chong, S. Lin, "Global stability of a commensal symbiosis model with Holling II functional response and feedback controls," Wseas Trans. Syst. Contr, vol.17, no.1, pp. 279-286, 2022.
[44] L. Y. Zanette, A. F. White, M. C. Allen, M. C., M. Clinchy, Perceived predation risk reduces the number of offspring songbirds produce per year Science, vol.334, no.6061 pp. 1398-1401, 2011.
[45] K. H. Elliott, G. S. Betini, D. R. Norris, "Experimental evidence for within- and cross-seasona effects of fear on survival and reproduction," Journal of Animal Ecology, vol.85, no.2, pp. 507-515, 2016.
[46] K. J. MacLeod, C. J. Krebs, R. Boonstra, et al. "Fear and lethality in snowshoe hares: the deadly effects of nonconsumptive predation risk," Oikos, vol. 127, no. 3, pp. 375-380, 2018.
[47] X. Wang, L. Zanette, X. Zou, "Modelling the fear effect in predator–prey interactions," Journal of Mathematical Biology, vol. 73, no. 5, pp. 1179-1204, 2016.
[48] A. J. Kashyap, D. Bhattacharjee, & H. K. Sarmah, " A fractional model in exploring the role of fear in mass mortality of pelicans in the Salton Sea," An International Journal of Optimization and Control: Theories & Applications (IJOCTA), vol. 11, no. 3, pp. 28–51, 2011.
[49] A. J. Kashyap, Q. Zhu, H. K. Sarmah, & D. Bhattacharjee, " Dynamical study of a predator–prey system with Michaelis–Menten type predator-harvesting," International Journal of Biomathematics, vol.16, no. 08, 2250135, 2023.
[50] M. Das, G. P. Samanta, "A prey-predator fractional order model with fear effect and group defence," International Journal of Dynamics and Control, vol. 9, no.1, pp. 334-349, 2021.

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