Indexed Fibonacci Arrays And Its Properties

Hannah Blasiyus, *Member, IAENG,* and D. K. Sheena Christy, *Member, IAENG,*

***Abstract*—n^{th} order Fibonacci word f_n is defined recursively by fixing the first two initial letters, namely f_0 and f_1. W.H. Chaun (1993) proposed a few ways of indexing the Fibonacci words with binary sequences in order to generate the Fibonacci words. Two-dimensional Fibonacci words (or Fibonacci arrays) were first defined in the year 2000 as an infinite sequence of arrays obtained recursively by fixing the four initial letters, namely $f_{0,0}, f_{0,1}, f_{1,0}$ and $f_{1,1}$. In 2021, few results on indexed Fibonacci and indexed involutive Fibonacci words have been studied. In this paper, we extend their work to Fibonacci arrays by defining a new way of indexing and proving results based on the indexing.**

***Index Terms*—Fibonacci words, Indexed Fibonacci words, Fibonacci arrays, Two-dimensional languages**

I. Introduction

FORMAL languages is the area of mathematics that serves as the basic tool for creating compilers. It has contributed to the growth of various other fields in the past few decades, such as computer networking, physics, biology, and so on. Various models proposed over the years, which emerged from the framework of image processing and pattern matching, can generalize formal languages to two dimensions [1], [2]. Two-dimensional patterns also appear in studies concerning combinatorics and other models of parallel computing.

The study of combinatorial properties - as they relate to formal languages - is the focus of the subject of combinatorics on words, which crosses the boundaries of computer science and mathematics. Finding patterns within sequences of symbols is the main focus of this area. The study of combinatorics on words originated in the early 1900s with the work of Axel Thue [3], who published papers on patterns and repetitions within words [4]. In 1983, Lothaire published his book 'Combinatorics on Words' [5]. Over the years, researchers have investigated numerous combinatorial features of words and arrays. To name a few recent publications in this area, Sasikala et al. [6] studied partial languages; Janaki and Arulprakasam discussed tiling systems in partial array languages in [7]; Kumari and Arulprakasam [8] investigated factors and subwords of rich partial words; and so on. The development of combinatorics on words and arrays has paved the way for significant advancements in automata theory. For example, John Kaspar et al. conducted research concerning lattice automata [9].

Knuth discovered the one-dimensional Fibonacci strings [10] to be the terms equivalent to the Fibonacci numbers given by $F_0 = 0, F_1 = 1$, and the recursion $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$, as an infinite sequence of strings obtained by fixing the first two initial letters, say $f_0 = p$ and $f_1 = q$, and recursively obtaining $f_{n+2} = f_{n+1} \bullet f_n$, for all $n \geq 1$, where $\bullet$ denotes word concatenation. Over the decades, numerous researchers have worked to uncover various combinatorial properties. One such work is by A. De. Luca [11].

W.F. Chuan defined a few new ways of indexing the Fibonacci words based on the nature of concatenation, which yielded [12]. Indexing the Fibonacci words paved the way for developing algorithms to generate them [13].

In 2000, Apostolico and Brimkov [14] defined the sequence of two-dimensional Fibonacci arrays as $f_{0,0} = p, f_{0,1} = q, f_{1,0} = r$ and $f_{1,1} = s$, where p, q, r and s are symbols from the finite alphabet Σ, and $f_{t,(j+1)} = f_{t,j} \oplus f_{t,(j-1)}$ and $f_{(i+1),t} = f_{i,t} \ominus f_{(i-1),t}$, for $t \geq 0$ and $i, j \geq 1$, where $\oplus$ denotes column concatenation and $\ominus$ denotes row concatenation.

Lila Kari et al. conducted speculative research on DNA computing, which led to the study of the involutive Fibonacci words in 2021 [15]. They have discussed indexed involutive Fibonacci words and studied some of their properties.

Involutive Fibonacci arrays were defined by Hannah Blasiyus and D.K. Sheena Christy in [16] and their combinatorial properties have been discussed in [17] and [18].

In this paper, we extend their work by defining a new method of indexing Fibonacci arrays and studying their properties.

We organise this paper as follows: Section II focuses on recalling several definitions from the literature that are essential for our study. In Section III, we define row-wise indexing and column-wise indexing of Fibonacci arrays. In Section IV, we study a few results based on the structure of arrays generated by both types of indices. Finally, we define symmetric Fibonacci arrays in Section V and discuss their decomposition into smaller arrays.

II. Preliminaries

Here we recall a few definitions required for our study. We refer the reader to [19] for the basic definitions.

Definition II.1. [20] A *picture* is an array or a two-dimensional string taken from any finite alphabet Σ. A collection of all such pictures or arrays is generally termed *two-dimensional language*, denoted by Σ^{**}.

Definition II.2. [20] If

$$P = \begin{array}{ccc} i_{11} & \dots & i_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ i_{m1} & \dots & i_{mn} \end{array},$$

and

Manuscript received October 12, 2023; revised May 06, 2024.

Hannah Blasiyus is a research scholar in the Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603203, India. (e-mail: hb3121@srmist.edu.in).

D. K. Sheena Christy is an Assistant Professor (Sr.G) in the Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603 203, India. (Corresponding author: e-mail: sheena.lesley@gmail.com).

$$Q = \begin{array}{ccc} j_{11} & \cdots & j_{1n'} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ j_{m'1} & \cdots & j_{m'n'} \end{array}$$

are arrays from Σ^{**}, then the *column concatenation* is defined if $m = m'$ as

$$P \varobar Q = \begin{array}{cccccc} i_{11} & \cdots & i_{1n} & j_{11} & \cdots & j_{1n'} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{m1} & \cdots & i_{mn} & j_{m'1} & \cdots & j_{m'n'} \end{array}$$

and the *row concatenation* is defined if $n = n'$ as

$$P \ominus Q = \begin{array}{ccc} i_{11} & \cdots & i_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ i_{m1} & \cdots & i_{mn} \\ j_{11} & \cdots & j_{1n'} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ j_{m'1} & \cdots & j_{m'n'} \end{array}$$

Definition II.3. [13], [14] *Indexing* of one-dimensional Fibonacci words is given as:

$$f_0 = p, f_1 = q$$

and

$$f_n^{t_1 t_2 t_3 \cdots t_{n-2}} = \begin{cases} f_{n-1}^{t_1 t_2 t_3 \cdots t_{n-3}} \bullet f_{n-2}^{t_1 t_2 t_3 \cdots t_{n-4}}, \text{if} t_{n-2} = 0 \\ f_{n-2}^{t_1 t_2 t_3 \cdots t_{n-4}} \bullet f_{n-1}^{t_1 t_2 t_3 \cdots t_{n-3}}, \text{if} t_{n-2} = 1 \end{cases}$$

Definition II.4. [15] A function that is its own inverse is called the *involution*. In other words, a function ϕ_1 such that ϕ_1^2 equals the identity is called an involution. i.e., $\phi_1(\phi_1(x)) = x$, for all x in the domain of ϕ_1.

Definition II.5. [16] For two-dimensional arrays, a function $\phi_1 : \Sigma^{**} \to \Sigma^{**}$ is a *morphism* if

1) $\phi_1(\epsilon) = \epsilon$, where ϵ denotes the empty array
2) $\phi_1(a \ominus b) = \phi_1(a) \ominus \phi_1(b)$ and
3) $\phi_1(a \varobar b) = \phi_1(a) \varobar \phi_1(b)$, for all $a, b \in \Sigma$, where $\ominus$ denotes row concatenation and $\varobar$ denotes column concatenation.

Definition II.6. [16] For two-dimensional arrays, a function $\phi_1 : \Sigma^{**} \to \Sigma^{**}$ is an *antimorphism* if

1) $\phi_1(\epsilon) = \epsilon$, where ϵ denotes the empty array
2) $\phi_1(a \ominus b) = \phi_1(b) \ominus \phi_1(a)$ and
3) $\phi_1(a \varobar b) = \phi_1(b) \varobar \phi_1(a)$, for all $a, b \in \Sigma$, where $\ominus$ denotes row concatenation and $\varobar$ denotes column concatenation.

Definition II.7. [16] A function $\phi_1 : \Sigma^{**} \to \Sigma^{**}$ is called a *morphic involution* on Σ^{**} if it is an involution on Σ extended to a morphism on Σ^{**}.

Definition II.8. [20] If

$$P = \begin{array}{ccc} i_{11} & \cdots & i_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ i_{m1} & \cdots & i_{mn} \end{array},$$

then the reflection of P about the rightmost vertical, denoted by $\tilde{P}$ is given by,

$$\tilde{P} = \begin{array}{ccc} i_{1n} & \cdots & i_{11} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ i_{mn} & \cdots & i_{m1} \end{array}$$

and the reflection of P about the base, denoted by $\hat{P}$ is given by,

$$\hat{P} = \begin{array}{ccc} i_{m1} & \cdots & i_{mn} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ i_{11} & \cdots & i_{1n} \end{array}$$

III. Indexing the Fibonacci arrays

Here we now define the new ways of indexing the two-dimensional Fibonacci arrays.

Notation. In this article, we use the notation $\begin{array}{cc} i_{0,0} & i_{0,1} \\ i_{1,0} & i_{1,1} \end{array}$ throughout to denote the first four positions of the array. i.e., the first two positions of the first row and the second row, respectively.

A. Row-indexing of Fibonacci arrays

Definition III.1. Let Σ be a finite alphabet over which the Fibonacci arrays $f_{m,n}$ are defined. Then the *Row-indexing of Fibonacci arrays* $f_{m,n}$ are defined recursively as follows:

1) $f_{0,0}^0 = w, f_{0,1}^{00} = x, f_{1,0}^1 = y, f_{1,1}^{11} = z$, where w, x, y and z are symbols from Σ with some of w, x, y and z might be identical
2) $f_{m,0}^m = f_{m-1,0}^{m-1} \ominus f_{m-2,0}^{m-2}$, for all $m \geq 2$
3) $f_{m,1}^{mm} = f_{m-1,1}^{(m-1)(m-1)} \ominus f_{m-2,1}^{(m-2)(m-2)}$, for all $m \geq 2$
4) For all $n \geq 2$,

$$f_{m,n}^{mmt_3t_4\ldots t_k 0} = f_{m,n-1}^{mmt_3t_4\ldots t_k} \varobar f_{m,n-2}^{mmt_3t_4\ldots t_{k-1}}$$

and

$$f_{m,n}^{mmt_3t_4\ldots t_k 1} = f_{m,n-2}^{mmt_3t_4\ldots t_{k-1}} \varobar f_{m,n-1}^{mmt_3t_4\ldots t_k}$$

where $t_i \in \{0, 1\}$, for $i \geq 3$.

The numerical sequence found in the exponent of each Fibonacci array is its *row-index*.

Remark 1. The length of the row index of $f_{m,n}$ is always $n + 1$.

Example III.1. The Fibonacci array corresponds to the initial array $\begin{array}{cc} a & b \\ c & d \end{array}$ with row number $m = 3$ and with the row index 33101 is

$$\begin{aligned} f_{3,4}^{33101} &= f_{3,2}^{331} \varobar f_{3,3}^{3310} = \{f_{3,0}^3 \varobar f_{3,1}^{33}\} \varobar \{f_{3,2}^{331} \varobar f_{3,1}^{33}\} \\ &= \{f_{3,0}^3 \varobar f_{3,1}^{33}\} \varobar \{f_{3,0}^3 \varobar f_{3,1}^{33} \varobar f_{3,1}^{33}\} \\ &= \left(f_{2,0}^2 \ominus f_{1,0}^1\right) \varobar \left(f_{2,1}^{22} \ominus f_{1,1}^{11}\right) \varobar \left(f_{2,0}^2 \ominus f_{1,0}^1\right) \\ &\quad \varobar \left(f_{2,1}^{22} \ominus f_{1,1}^{11}\right) \varobar \left(f_{2,1}^{22} \ominus f_{1,1}^{11}\right) \\ &= \left(f_{1,0}^1 \ominus f_{0,0}^0 \ominus f_{1,0}^1\right) \varobar \left(f_{1,1}^{11} \ominus f_{0,1}^{00} \ominus f_{1,1}^{11}\right) \\ &\quad \varobar \left(f_{1,1}^{11} \ominus f_{0,1}^{00} \ominus f_{1,1}^{11}\right) \varobar \left(f_{1,0}^1 \ominus f_{0,0}^0 \ominus f_{1,0}^1\right) \end{aligned}$$

$$= \begin{array}{ccccc} c & d & c & d & d \\ a & b & a & b & b \\ c & d & c & d & d \end{array}$$

B. Column-indexing of Fibonacci arrays

Definition III.2. Let Σ be an alphabet. Then the *Column-indexing of Fibonacci arrays* $f_{m,n}$ are defined recursively as follows:

1) $f_{0,0}^{0} = w, f_{0,1}^{1} = x, f_{1,0}^{00} = y, f_{1,1}^{11} = z$, where w, x, y and z are symbols from Σ with some of w, x, y and z might be identical
2) $f_{0,n}^{n} = f_{0,n-1}^{n-1} \oplus f_{0,n-2}^{n-2}$, for all $n \geq 2$
3) $f_{1,n}^{nn} = f_{1,n-1}^{(n-1)(n-1)} \oplus f_{1,n-2}^{(n-2)(n-2)}$, for all $n \geq 2$
4) For all $m \geq 2$,

$$f_{m,n}^{nnt_3t_4\ldots t_k0} = f_{m-1,n}^{nnt_3t_4\ldots t_k} \ominus f_{m-2,n}^{nnt_3t_4\ldots t_{k-1}}$$

and

$$f_{m,n}^{nnt_3t_4\ldots t_k1} = f_{m-2,n}^{nnt_3t_4\ldots t_{k-1}} \ominus f_{m-1,n}^{nnt_3t_4\ldots t_k},$$

where $t_i \in \{0, 1\}$, for $i \geq 3$.

The numerical sequence found in the exponent of each Fibonacci array is its *column-index*.

Remark 2. The length of the column index of $f_{m,n}$ is always $m + 1$.

Example III.2. The Fibonacci array corresponding to the initial symbols $\begin{matrix} a & b \\ c & d \end{matrix}$ with column number $n = 4$ and with the column-index 4410 is

$$\begin{aligned}
&f_{3,4}^{4410} = f_{2,4}^{441} \ominus f_{1,4}^{44} = \{f_{0,4}^{4} \ominus f_{1,4}^{44}\} \ominus f_{1,4}^{44} \\
&= \{f_{0,3}^{3} \oplus f_{0,2}^{2}\} \ominus \{f_{1,3}^{33} \oplus f_{1,2}^{22}\} \ominus \{f_{1,3}^{33} \oplus f_{1,2}^{22}\} \\
&= \left(f_{0,2}^{2} \oplus f_{0,1}^{1} \oplus f_{0,1}^{1} \oplus f_{0,0}^{0}\right) \ominus \left(f_{1,2}^{22} \oplus f_{1,1}^{11} \oplus f_{1,1}^{11} \oplus f_{1,0}^{00}\right) \\
&\qquad \ominus \left(f_{1,2}^{22} \oplus f_{1,1}^{11} \oplus f_{1,1}^{11} \oplus f_{1,0}^{00}\right) \\
&= \left(f_{0,1}^{1} \oplus f_{0,0}^{0} \oplus f_{0,1}^{1} \oplus f_{0,1}^{1} \oplus f_{0,0}^{0}\right) \\
&\qquad \ominus \left(f_{1,1}^{11} \oplus f_{1,0}^{00} \oplus f_{1,1}^{11} \oplus f_{1,1}^{11} \oplus f_{1,0}^{00}\right) \\
&\qquad \ominus \left(f_{1,1}^{11} \oplus f_{1,0}^{00} \oplus f_{1,1}^{11} \oplus f_{1,1}^{11} \oplus f_{1,0}^{00}\right)
\end{aligned}$$

$$= \begin{matrix} b & a & b & b & a \\ d & c & d & d & c \\ d & c & d & d & c \end{matrix}$$

IV. Some results on Indexed Fibonacci arrays

In this section, we study a few results based on the row and column indexing of the Fibonacci arrays.

Theorem IV.1. *Let ϕ_1 be a morphic involution on Σ^{**}, where $\Sigma = \{a, b, c, d\}$. Then, for all $n \geq 2$, we have,*

$$\phi_1\left(f_{m,n}^{mmt_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f_{m,n}^{mmt_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix}$$

where $m \geq 0, t_i's \in \{0, 1\}, 3 \leq i \leq n + 1$.

Proof: We prove this result by strong induction on m. At first, we take $m = 0$ or $m = 1$. As this result is proven for the case of one-dimension in [15], we say that the basis for m is true, where $n \geq 2$. Next, we assume that the result holds for $2 \leq m \leq i - 1$.

We have to prove that

$$\phi_1\left(f_{i,n}^{iit_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f_{i,n}^{iit_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix}.$$

Consider the LHS, $\phi_1\left(f_{i,n}^{iit_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

By Definition III.1, we have

$$\phi_1\left(f_{i,n}^{iit_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= \begin{cases} \phi_1\left(f_{i,n-1}^{iit_3t_4\ldots t_n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus f_{i,n-2}^{iit_3t_4\ldots t_{n-1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \text{if } t_{n+1} = 0 \\ \phi_1\left(f_{i,n-2}^{iit_3t_4\ldots t_{n-1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus f_{i,n-1}^{iit_3t_4\ldots t_n} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \text{if } t_{n+1} = 1 \end{cases}$$

$$= \begin{cases} \phi_1\left(f_{i,n-1}^{iit_3t_4\ldots t_n} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \oplus \phi_1\left(f_{i,n-2}^{iit_3t_4\ldots t_{n-1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \text{if } t_{n+1} = 0 \\ \phi_1\left(f_{i,n-2}^{iit_3t_4\ldots t_{n-1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \oplus \phi_1\left(f_{i,n-1}^{iit_3t_4\ldots t_n} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \text{if } t_{n+1} = 1 \end{cases}$$

since ϕ_1 is a morphic involution on Σ^{**}. By recursively reducing the above equation to an equation in terms of $f_{i,0}$ and $f_{i,1}$, we can apply Definition III.1 appropriately, and with the help of the induction hypothesis, we conclude that the result holds for $m = i$. (i.e.,)

$$\phi_1\left(f_{i,n}^{iit_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f_{i,n}^{iit_3t_4\ldots t_nt_{n+1}} \begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix}$$

By mathematical induction, the result holds for all $m \geq 0$. ∎

Example IV.1. From Example 3.1,

$$f_{3,4}^{33101} = \begin{matrix} c & d & c & d & d \\ a & b & a & b & b \\ c & d & c & d & d \end{matrix}$$

Let ϕ_1 be a morphic involution on Σ^{**}, where $\Sigma = \{a, b, c, d\}$, defined by $\phi_1(a) = c, \phi_1(b) = d, \phi_1(c) = a$

and $\phi_1(d) = b$. Then

$$\phi_1\left(f_{3,4}^{33101}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \phi_1\begin{pmatrix} c & d & c & d & d \\ a & b & a & b & b \\ c & d & c & d & d \end{pmatrix}$$
$$= \begin{matrix} \phi_1(c) & \phi_1(d) & \phi_1(c) & \phi_1(d) & \phi_1(d) \\ \phi_1(a) & \phi_1(b) & \phi_1(a) & \phi_1(b) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) & \phi_1(c) & \phi_1(d) & \phi_1(d) \end{matrix}$$

which is same as $f_{3,4}^{33101}\begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix}$, satisfying the theorem.

Note: In Theorem IV.1 we have considered ϕ_1 to be a morphic involution. However, this result fails when ϕ_1 is an antimorphic involution.

Example IV.2. The Fibonacci array corresponding to the initial array $\begin{matrix} a & b \\ c & d \end{matrix}$ with row index 2201 is given by $f_{2,3}^{2201}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{matrix} d & d & c \\ b & b & a \end{matrix}$. If ϕ_1 is an antimorphic involution over $\{a, b, c, d\}$ given by $\phi_1(a) = c$ and $\phi_1(b) = d$, then $\phi_1\left(f_{2,3}^{2201}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{matrix} c & d & d \\ a & b & b \end{matrix}$. But we find that $f_{2,3}^{2201}\begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix} = \begin{matrix} b & b & a \\ d & d & c \end{matrix}$ and they are not the same.

Theorem IV.2. *Let ϕ_1 be a morphic involution on Σ^{**}, where $\Sigma = \{a, b, c, d\}$. Then, for all $m \geq 2$, we have,*

$$\phi_1\left(f_{m,n}^{nnt_3t_4\ldots t_mt_{m+1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f_{m,n}^{nnt_3t_4\ldots t_mt_{m+1}}\begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix},$$

where $n \geq 0, t_i's \in \{0, 1\}, 3 \leq i \leq m + 1$.

Proof: We prove this result by strong induction on n. At first, we take $n = 0$ or $n = 1$. As this result is proven for the case of one-dimension in [15], we say that the basis for n is true, where $m \geq 2$. Next, we assume that the result holds for $2 \leq n \leq i - 1$. We have to prove that

$$\phi_1\left(f_{m,i}^{iit_3t_4\ldots t_mt_{m+1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f_{m,i}^{iit_3t_4\ldots t_mt_{m+1}}\begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix}.$$

Consider the LHS, $\phi_1\left(f_{m,i}^{iit_3t_4\ldots t_mt_{m+1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

By Definition III.2, we have

$$= \begin{cases} \phi_1\left(f_{m-1,i}^{iit_3t_4\ldots t_m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \ominus \\ f_{m-2,i}^{iit_3t_4\ldots t_{m-1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Bigg), \text{if } t_{m+1} = 0 \\ \phi_1\left(f_{m-2,i}^{iit_3t_4\ldots t_{m-1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \ominus \\ f_{m-1,i}^{iit_3t_4\ldots t_m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Bigg), \text{if } t_{m+1} = 1 \end{cases}$$

$$= \begin{cases} \phi_1\left(f_{m-1,i}^{iit_3t_4\ldots t_m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \ominus \\ \phi_1\left(f_{m-2,i}^{iit_3t_4\ldots t_{m-1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \text{if } t_{m+1} = 0 \\ \phi_1\left(f_{m-2,i}^{iit_3t_4\ldots t_{m-1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \ominus \\ \phi_1\left(f_{m-1,i}^{iit_3t_4\ldots t_m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \text{if } t_{m+1} = 1 \end{cases}$$

since ϕ_1 is a morphic involution on Σ^{**}. By recursively reducing the above equation to an equation in terms of $f_{0,i}$ and $f_{1,i}$, we can apply Definition III.2 appropriately, and with the help of the induction hypothesis, we conclude that the result holds for $n = i$. (i.e.,)

$$\phi_1\left(f_{m,i}^{iit_3t_4\ldots t_mt_{m+1}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f_{m,i}^{iit_3t_4\ldots t_mt_{m+1}}\begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix}$$

Thus, by principle of mathematical induction, the result holds for all $n \geq 0$. ∎

Example IV.3. From Example 3.2,

$$f_{3,4}^{4410} = \begin{matrix} b & a & b & b & a \\ d & c & d & d & c \\ d & c & d & d & c \end{matrix}$$

Let ϕ_1 be a morphic involution on Σ^{**}, where $\Sigma = \{a, b, c, d\}$, defined by $\phi_1(a) = b, \phi_1(b) = a, \phi_1(c) = d$ and $\phi_1(d) = c$. Then

$$\phi_1\left(f_{3,4}^{4410}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \phi_1\begin{pmatrix} b & a & b & b & a \\ d & c & d & d & c \\ d & c & d & d & c \end{pmatrix}$$
$$= \begin{matrix} \phi_1(b) & \phi_1(a) & \phi_1(b) & \phi_1(b) & \phi_1(a) \\ \phi_1(d) & \phi_1(c) & \phi_1(d) & \phi_1(d) & \phi_1(c) \\ \phi_1(d) & \phi_1(c) & \phi_1(d) & \phi_1(d) & \phi_1(c) \end{matrix}$$

which is same as $f_{3,4}^{4410}\begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix}$, satisfying the theorem.

Note: In Theorem IV.2 we have considered ϕ_1 to be a morphic involution. However, this result fails when ϕ_1 is an antimorphic involution.

Example IV.4. The Fibonacci array corresponding to the initial array $\begin{matrix} a & b \\ c & d \end{matrix}$ with column index 2201 is given by $f_{3,2}^{2201}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{matrix} d & c \\ d & c \\ b & a \end{matrix}$. If ϕ_1 is an antimorphic involution over $\{a, b, c, d\}$ given by $\phi_1(a) = c$ and $\phi_1(b) = d$, then $\phi_1\left(f_{3,2}^{2201}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{matrix} c & d \\ a & b \\ a & b \end{matrix}$. But we find that $f_{3,2}^{2201}\begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_1(c) & \phi_1(d) \end{pmatrix} = \begin{matrix} b & a \\ b & a \\ d & c \end{matrix}$ and they are not the same.

Theorem IV.3. *For all $m \geq 0$ and $n \geq 2$, the row-indexed array*

$$f_{m,n}^{mmt_3t_4\cdots t_{n+1}} = \widetilde{f_{m,n}^{mmr_3r_4\cdots r_{n+1}}},$$

where $r_i = 1-t_i, 2 \leq i \leq n+1$ and $\widetilde{f_{m,n}^{mmr_3r_4\cdots r_{n+1}}}$ denotes the reflection of $f_{m,n}^{mmr_3r_4\cdots r_{n+1}}$ about its rightmost vertical.

Proof: We prove this by the method of induction on n. Consider the basis for n, $n = 2$. Then,

$$f_{m,2}^{mmt_3} = \begin{cases} f_{m,1}^{mm} \oplus f_{m,0}^{m}, & \text{if } t_3 = 0 \\ f_{m,0}^{m} \oplus f_{m,1}^{mm}, & \text{if } t_3 = 1 \end{cases}$$

$$= \begin{cases} \widetilde{f_{m,0}^{m} \oplus f_{m,1}^{mm}}, & \text{if } r_3 = 1 \\ \widetilde{f_{m,1}^{mm} \oplus f_{m,0}^{m}}, & \text{if } r_3 = 0 \end{cases}$$

$$= \begin{cases} \widetilde{f_{m,2}^{mmr_3}}, & \text{if } r_3 = 1 \\ \widetilde{f_{m,2}^{mmr_3}}, & \text{if } r_3 = 0 \end{cases}$$

i.e., $f_{m,2}^{mmt_3} = \widetilde{f_{m,2}^{mmr_3}}$ and therefore the basis for n is proved true.Assume that for $3 \leq j \leq i-1$ the result holds. Then,

$$f_{m,i}^{mmt_3\cdots t_{i+1}} = \begin{cases} f_{m,i-1}^{mmt_3\cdots t_i} \oplus f_{m,i-2}^{mmt_3\cdots t_{i-1}}, & \text{if } t_{i+1} = 0 \\ f_{m,i-2}^{mmt_3\cdots t_{i-1}} \oplus f_{m,i-1}^{mmt_3\cdots t_i}, & \text{if } t_{i+1} = 1 \end{cases}$$

$$= \begin{cases} \widetilde{f_{m,i-1}^{mmr_3\cdots r_i}} \oplus \widetilde{f_{m,i-2}^{mmr_3\cdots r_{i-1}}}, & \text{if } r_{i+1} = 1 \\ \widetilde{f_{m,i-2}^{mmr_3\cdots r_{i-1}}} \oplus \widetilde{f_{m,i-1}^{mmr_3\cdots r_i}}, & \text{if } r_{i+1} = 0 \end{cases}$$

$$= \begin{cases} \widetilde{f_{m,i}^{mmr_3\cdots r_{i+1}}}, & \text{if } r_{i+1} = 1 \\ \widetilde{f_{m,i}^{mmr_3\cdots r_{i+1}}}, & \text{if } r_{i+1} = 0 \end{cases}$$

i.e., $f_{m,i}^{mmt_3\cdots t_{i+1}} = \widetilde{f_{m,i}^{mmr_3\cdots r_{i+1}}}$. Hence, by the principle of mathematical induction, the result holds for all $n \geq 2$. ∎

Example IV.5. *We find that $f_{4,5}^{440101}$ is the row-indexed array given by*

$$\begin{array}{cccccccc} d & d & c & d & d & c & d & c \\ b & b & a & b & b & a & b & a \\ d & d & c & d & d & c & d & c \\ d & d & c & d & d & c & d & c \\ b & b & a & b & b & a & b & a \end{array}$$

and $f_{4,5}^{441010}$ is the row-indexed array given by

$$\begin{array}{cccccccc} c & d & c & d & d & c & d & d \\ a & b & a & b & b & a & b & b \\ c & d & c & d & d & c & d & d \\ c & d & c & d & d & c & d & d \\ a & b & a & b & b & a & b & b \end{array},$$

which is the reflection of $f_{4,5}^{440101}$ about the rightmost vertical.

Theorem IV.4. *For all $n \geq 0$ and $m \geq 2$, the column-indexed array*

$$f_{m,n}^{nnt_3t_4\cdots t_{n+1}} = \widehat{f_{m,n}^{nnr_3r_4\cdots r_{n+1}}},$$

where $r_i = 1-t_i, 3 \leq i \leq m+1$ and $\widehat{f_{m,n}^{mmr_3r_4\cdots r_{n+1}}}$ denotes the reflection of $f_{m,n}^{mmr_3r_4\cdots r_{n+1}}$ about the base.

Proof: We prove this by the method of induction on m. Consider the basis for m, $m = 2$. Then,

$$f_{2,n}^{nnt_3} = \begin{cases} f_{1,n}^{nn} \ominus f_{0,n}^{n}, & \text{if } t_3 = 0 \\ f_{0,n}^{n} \ominus f_{1,n}^{nn}, & \text{if } t_3 = 1 \end{cases}$$

$$= \begin{cases} \widehat{f_{0,n}^{n} \ominus f_{1.n}^{nn}}, & \text{if } r_3 = 1 \\ \widehat{f_{1,n}^{nn} \ominus f_{0,n}^{n}}, & \text{if } r_3 = 0 \end{cases}$$

$$= \begin{cases} \widehat{f_{2,n}^{nnr_3}}, & \text{if } r_3 = 1 \\ \widehat{f_{2,n}^{nnr_3}}, & \text{if } r_3 = 0 \end{cases}$$

i.e., $f_{2,n}^{nnt_3} = \widehat{f_{2,n}^{nnr_3}}$ and therefore the basis for n is proved to be true. Assume that for $3 \leq j \leq i-1$ the result holds. Then,

$$f_{i,n}^{nnt_3\cdots t_{i+1}} = \begin{cases} f_{i-1,n}^{nnt_3\cdots t_i} \ominus f_{i-2,n}^{nnt_3\cdots t_{i-1}}, & \text{if } t_{i+1} = 0 \\ f_{i-2,n}^{nnt_3\cdots t_{i-1}} \ominus f_{i-1,n}^{nnt_3\cdots t_i}, & \text{if } t_{i+1} = 1 \end{cases}$$

$$= \begin{cases} \widehat{f_{i-1,n}^{nnr_3\cdots r_i}} \ominus \widehat{f_{i-2,n}^{nnr_3\cdots r_{i-1}}}, & \text{if } r_{i+1} = 1 \\ \widehat{f_{i-2,n}^{nnr_3\cdots r_{i-1}}} \ominus \widehat{f_{i-1,n}^{nnr_3\cdots r_i}}, & \text{if } r_{i+1} = 0 \end{cases}$$

$$= \begin{cases} \widehat{f_{i,n}^{nnr_3\cdots r_{i+1}}}, & \text{if } r_{i+1} = 1 \\ \widehat{f_{i,n}^{nnr_3\cdots r_{i+1}}}, & \text{if } r_{i+1} = 0 \end{cases}$$

i.e., $f_{i,n}^{nnt_3\cdots t_{i+1}} = \widehat{f_{i,n}^{nnr_3\cdots r_{i+1}}}$. Hence, by the principle of mathematical induction, the result holds for all $m \geq 2$. ∎

Example IV.6. We find that $f_{3,4}^{4410}$ is the column-indexed array given by $\begin{array}{ccccc} b & a & b & b & a \\ d & c & d & d & c \\ d & c & d & d & c \end{array}$ and $f_{3,4}^{4401}$ is the column-indexed array given by $\begin{array}{ccccc} d & c & d & d & c \\ d & c & d & d & c \\ b & a & b & b & a \end{array}$, which is the reflection of $f_{3,4}^{4410}$ about the base.

Theorem IV.5. *The indexed arrays $f_{m,n}^{\lambda 00}$ and $f_{m,n}^{\lambda 11}$ are equal for the same initial symbols, where $\lambda = mmt_3t_4\cdots t_{n-1}$ or $nnt_3t_4\cdots t_{m-1}$ denotes the preceding digits of the index.*

Proof: In the case of row indexing,

$$\begin{aligned} f_{m,n}^{\lambda 00} &= f_{m,n}^{mmt_3t_4\cdots t_{n-1}00} \\ &= f_{m,n-1}^{mmt_3t_4\cdots t_{n-1}0} \oplus f_{m,n-2}^{mmt_3t_4\cdots t_{n-1}} \\ &= f_{m,n-2}^{mmt_3t_4\cdots t_{n-1}} \oplus f_{m,n-3}^{mmt_3t_4\cdots t_{n-2}} \oplus f_{m,n-2}^{mmt_3t_4\cdots t_{n-1}} \\ &= f_{m,n-2}^{\lambda} \oplus f_{m,n-1}^{\lambda 1} = f_{m,n}^{\lambda 11} \end{aligned}$$

Similarly, we can prove it for column-indexed Fibonacci arrays. ∎

Example IV.7. Consider the row-indexed array

$$f_{2,3}^{2200}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We find this to be equal to $\begin{array}{ccc} d & c & d \\ b & a & b \end{array}$, which is the same as $f_{2,3}^{2211}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, satisfying the theorem.

Remark 3. The index for the standard Fibonacci arrays $f_{m,n}$ is given by $f_{m,n}^{mm00\cdots 0}, n \geq 2$ or $f_{m,n}^{nn00\cdots 0}, m \geq 2$.

Remark 4. The index for the reverse Fibonacci arrays $f'_{m,n}$ is given by $f_{m,n}^{mm11\cdots 1}, n \geq 2$ or $f_{m,n}^{nn11\cdots 1}, m \geq 2$.

V. Symmetric Fibonacci arrays

In this section we define symmetric Fibonacci arrays and study their behaviour.

Definition V.1. The standard Fibonacci arrays given by $f_{m,n}^{mm00\cdots 0}$ or $f_{m,n}^{nn00\cdots 0}$ with $f_{0,0}^{0} = f_{1,1}^{11} = w$ and $f_{0,1}^{1} =$

$f_{1,0}^{00} = x$ give rise to square standard Fibonacci arrays $f_{m,m}, m \geq 2$, whose transpose matches with the arrays themselves. Such arrays are called as *Standard symmetric Fibonacci arrays*, denoted by $st.sym(f_{m,m})$.

Definition V.2. The reverse Fibonacci arrays given by $f_{m,n}^{mm11\cdots1}$ or $f_{m,n}^{nn11\cdots1}$ with $f_{0,0}^{0} = f_{1,1}^{11} = w$ and $f_{0,1}^{1} = f_{1,0}^{00} = x$ give rise to square reverse Fibonacci arrays $f'_{m,m}, m \geq 2$, whose transpose matches with the arrays themselves. Such arrays are called as *Reverse symmetric Fibonacci arrays*, denoted by $rev.sym(f_{m,m})$.

Example V.1.

$$st.sym(f_{4,4}) = f_{4,4}^{44000} = \begin{matrix} w & x & w & w & x \\ x & w & x & x & w \\ w & x & w & w & x \\ w & x & w & w & x \\ x & w & x & x & w \end{matrix}$$

and

$$rev.sym(f_{3,3}) = f_{3,3}^{4411} = \begin{matrix} w & x & w \\ x & w & x \\ w & x & w \end{matrix}$$

.

Theorem V.1. The standard symmetric Fibonacci words $st.sym(f_{m,m})$ with appropriate index can be decomposed as $(P \ominus Q) \oplus (R \ominus S)$, where P, Q, R and S possess appropriate indices, $P = st.sym(f_{m-1,m-1})$ and $S = st.sym(f_{m-2,m-2})$ and $Q = R^T$, the transpose of R.

Proof: Since $f_{m,m}$ can be expressed as $f_{m,m} = f_{m,m-1} \oplus f_{m,m-2}$ which in turn equals $[f_{m-1,m-1} \ominus f_{m-2,m-1}] \oplus [f_{m-1,m-2} \ominus f_{m-2,m-2}]$ and since $f_{m,m}$ is symmetric in nature the result holds. ■

We now give a similar result for reverse Fibonacci arrays.

Theorem V.2. The reverse symmetric Fibonacci words $rev.sym(f_{m,m})$ with appropriate index can be decomposed as $(P \ominus Q) \oplus (R \ominus S)$, where P, Q, R and S possess appropriate indices, $P = rev.sym(f_{m-2,m-2})$ and $S = rev.sym(f_{m-1,m-1+})$ and $Q = R^T$, the transpose of R.

Proof: Since $f'_{m,m}$ can be expressed as $f'_{m,m} = f'_{m,m-2} \oplus f'_{m,m-1}$ which in turn equals $[f'_{m-2,m-2} \ominus f'_{m-1,m-2}] \oplus [f'_{m-2,m-1} \ominus f'_{m-1,m-1}]$ and since $f'_{m,m}$ is symmetric in nature the result holds. ■

VI. Conclusion

In this article, we have defined a new method of row indexing and column indexing for Fibonacci arrays, which helps in generating the Fibonacci arrays just with the help of their indices. Also, we have proved some of the properties of the indexed Fibonacci arrays. Further we have defined symmetric Fibonacci arrays and discussed a few of their properties. Future works will focus on studying other combinatorial properties based on the indexing of Fibonacci arrays and the generalizing the same to any two-dimensional arrays.

References

[1] Giammarresi, D., Restivo, A. "Recognizable picture languages", International Journal of Pattern Recognition and Artificial Intelligence, vol. 6, pp. 241-256, 1992.

[2] Giammarresi, D., Restivo, A. "Two-Dimensional languages", In: Rozenberg, G., Salomaa, A. (eds) Handbook of Formal Languages. Springer, Berlin, Heidelberg, 1997.

[3] Thue, A. "Über Unendliche Zeichenreihen", Norske Vid Selsk. Skr. I Mat-Nat Kl.(Christiana), vol. 7, pp. 1-22, 1906.

[4] Trygve Nagell, Atle Selberg, Sigmund Selberg, Knut Thalberg "Selected Mathematical Papers of Axel Thue", Universitetsforlaget, Oslo, 1977.

[5] Lothaire, M. "Combinatorics on Words", Encyclopedia of Mathematics and its Applications, Addison - Wesley, United States, 1983.

[6] Sasikala, K., Rajkumar Dare, V., D.G. Thomas "Learning of Partial Languages", Engineering Letters, vol. 14 no. 2, pp. 72-80, 2007.

[7] Janaki, K., Arulprakasam, R. "Tiling systems and domino systems for partial array languages", IAENG International Journal of Applied Mathematics, vol. 52, no. 2, pp. 418-425, 2022.

[8] Krishna Kumari, R., Arulprakasam, R. "Factors and Subwords of Rich Partial Words", IAENG International Journal of Applied Mathematics, vol. 53, no. 2, pp. 534-539, 2023.

[9] A. John Kaspar, D.K. Sheena Christy, D.G. Thomas "Lattice Regular Grammar-Automata", IAENG International Journal of Applied Mathematics, vol. 52, no.4, pp. 1123-1129, 2022.

[10] Knuth, D. E. "The Art of Computer Programming", "Volume 1: Fundamental Algorithms, 1^{st} edition, Addison - Wesley, United States", 1968.

[11] Aldo de Luca. "A Combinatorial property of the Fibonacci words", Information Processing Letters, vol.12, no. 4, pp. 193-195, 1981.

[12] Chuan, W-F. "Fibonacci words", The Fibonacci Quarterly, vol. 30, no. 1, pp. 68-76, 1992.

[13] Chuan, W-F. "Generating Fibonacci words", The Fibonacci Quarterly, vol. 33, no. 2, pp. 104-112, 1995.

[14] Apostolico, A., Brimkov, V.E. "Fibonacci arrays and their two – dimensional repetitions", Theoretical Computer Science, vol. 237, pp. 263-273, 2000.

[15] Kari, L., Kulkarni, M. S., Mahalingam, K., Wang, Z. "Involutive Fibonacci Words", Journal of Automata, Languages and Combinatorics, vol. 26, pp. 255-280, 2021.

[16] Hannah Blasiyus, D.K. Sheena Christy "On Involutive Co-Fibonacci words and Involutive Fibonacci Arrays", Communicated.

[17] Hannah Blasiyus, D. K. Sheena Christy "Combinatorial Properties of Involutive Fibonacci Arrays", IAENG International Journal of Computer Science, vol. 51, no. 5, pp. 553-561, 2024.

[18] Hannah Blasiyus, D.K. Sheena Christy "Palindromes in Involutive Fibonacci Arrays", Communicated.

[19] Linz, P., Rodger, S. H. "An Introduction to Formal languages and Automata", 7^{th} Edition, Sudbury, Massachusetts, Jones & Bartlett, 2023.

[20] Siromoney, G., Siromoney, R., Krithivasan, K. "Abstract Families of Matrices and Picture Languages", Computer Graphics and Image Processing, vol. 1, pp. 284-307, 1972.

[21] Siromoney, G., Siromoney, R., Krithivasan, K. "Picture Languages with Array Rewriting Rules", Information and Control, vol. 20, pp. 447-470, 1973.