The Influence of Fear Effect to the Dynamic Behaviors of Lotka-Volterra Ammensalism Model

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***Abstract*—This paper presents a study on a Lotka-Volterra ammensalism model that incorporates the fear effect, which can potentially decrease the birth rate and raise the mortality rate of the species. For the autonomous case, equilibrium points' local and global stability are discussed. For the nonautonomous case, sufficient conditions which ensure the persistence and extinction, and global asymptotic stability of the positive solutions are obtained, respectively. The study has shown that with the increase of the fear effect, the final density of the affected population will decrease, and when the fear effect is large enough, it will cause population's extinction.**

***Index Terms*—Fear effect; Ammensalism; Local stability; Global stability; Persistence; Extinction; Global asymptotically stability**

I. Introduction

It is rare to see a single species that can live independently without relationships with other natural populations. Amensalism is a biological phenomenon characterized by the interaction between two species, whereby one species experiences limitations and constraints, while the other species remains unaffected. Xi, Griffin, and Sun[1] pointed out that in a Tibetan alpine meadow, grasshoppers and grassland caterpillars constitute an amensal relationship. Gómez and González-Megías[2] pointed out that the Spanish ibex and the weevil also forms an amensalism relationship. In recent years, the research on the amensalism model has attracted great attention from scholars ([3]-[50]). For example, in [16], a detailed analysis of the amensalism system with the Beddington-DeAngelis functional response and the second population with the Allee effect was carried out, and it was proved that with the change of the system parameters, the system has dynamics such as saddle node bifurcation and transcritical bifurcation. The author also gave the global phase diagram of the system; Zhou, Chen, and Lin[30] argued that the discrete model is more appropriate when the population size is small, or the population intergenerational is obvious, so they proposed a discrete amensalism system with the Beddington–DeAngelis functional response and Allee effect for the unaffected species, their research shows that the Allee effect can enhance the stability of the system at this time, and as the parameters change, the system can have various codimension one and codimension two bifurcations, including transcritical bifurcations, pitchfork bifurcations, folding bifurcations, flipping bifurcations, and 1:2 strong resonance bifurcations, whose dynamic behaviors is far more complex than that of continuum systems.

Noticed that to this day, no scholars have considered the influence of fear on the amensalism system. Xi, Griffin, and Sun[1] pointed out grassland caterpillars and grasshoppers are two of the main herbivorous insects. Among them, the grasshoppers "no consciousness" disturbance (natural jumping) severely reduced the feeding time of grassland caterpillars, slowing down its development rate, and eventually lead to a decline in the number of eggs laid by female caterpillars, the presence of grasshoppers had a substantial impact on various aspects of caterpillar behavior and development, including eating patterns, growth rate, survival rates, reproductive efforts, and the timing of transformation. That is to say, the fear of grasshoppers eventually leads to a decrease in the birth rate and an increase in the death rate of grassland caterpillars. It is necessary to propose suitable mathematical modeling to describe such a fact. However, so far, no scholars have proposed or studied the amensalism model with the fear effect. This paper aims to establish the Lotka-Volterra model with the fear effect on the affected species, and find out how the fear effect affects the dynamic behavior of the amensalism model.

The classic two-species amensalism model can be expressed as

$$\begin{aligned} \frac{dx}{dt} &= x(a_1 - b_1 x - c_1 y), \\ \frac{dy}{dt} &= y(a_2 - c_2 y), \end{aligned} \tag{1}$$

Where $x \geq 0, y \geq 0, a_1, a_2, b_1, c_2, c_1$ are all normal numbers, where $a_i, i = 1, 2$ represent the intrinsic growth rates of the first and second populations respectively , b_i represents the intraspecific competition coefficient of the two populations, and c_1 represents the influence coefficient of the second population on the first population. Zhu and Chen [12] studied the stability of each equilibrium point of the system (1) and the trajectory of the system using vector field analysis. The author obtained the following results:

Theorem A. If

$$\frac{a_1}{c_1} > \frac{a_2}{c_2} \tag{2}$$

holds, the unique positive equilibrium point $A(\frac{a_1c_2 - c_1a_2}{b_1c_2}, \frac{a_2}{c_2})$ of the system (1) is globally stable. If

$$\frac{a_1}{c_1} < \frac{a_2}{c_2} \tag{3}$$

holds, then the boundary equilibrium point $B(0, \frac{a_2}{c_2})$ of the system (1) is globally stable.

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In the above system (1), a_1 represents the intrinsic growth rate of the first population, $a_1 = e_1 - e_2$, where e_1 is the birth rate and e_2 is the death rate. As mentioned earlier, the birth rate and death rate of the first population will be negatively affected by the second species, so it is not enough to assume that e_1 and e_2 are constants in the system (1). To be precise, we propose the following model:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{e_1}{1+k_1y} - (1+k_2y)e_2 - b_1x - c_1y\Big), \\ \frac{dy}{dt} &= y\Big(a_2 - c_2y\Big). \end{aligned} \tag{4}$$

Here, we use $\frac{1}{1+k_1y}$ and $1+k_2y$ to represent the fear effect, and $k_i, i = 1, 2$ to represent the fear factor. When $k_i = 0$, that is, when the first population is not affected by fear, system (4) degenerates into system (1). The purpose of this paper is to study the dynamic behaviors of the system (4), and to find out the influence of the fear term $\frac{1}{1+k_1y}$ and $1+k_2y$.

The subsequent sections of the paper are organized in the following manner: In the subsequent part, we shall examine the presence and regional stability of the equilibria of the system. Following this, in part III, we will analyze the equilibria's global stability characteristic. Section IV of this study examines the persistent, extinct, and global stability characteristics of the nonautonomous situation. Two specific instances are presented, along with numerical simulations, to demonstrate the practicality of the primary findings in Section V. The present paper concludes with a concise discussion.

II. Local Stability of the Equilibria

The equilibrium points of the system (4) satisfy the equation

$$\begin{aligned} x\Big(\frac{e_1}{1+k_1y} - e_2(1+k_2y) - b_1x - c_1y\Big) &= 0, \\ y\Big(a_2 - c_2y\Big) &= 0. \end{aligned} \tag{5}$$

The calculation shows that the system always has boundary equilibrium points $A_1(0,0)$, $A_2(0, \frac{a_2}{c_2})$, in addition, if

$$e_1 > e_2 \tag{6}$$

holds, there is a boundary equilibrium point $A_3(\frac{e_1-e_2}{b_1}, 0)$, if

$$e_1c_2^2 > (a_2k_1 + c_2)(e_2k_2a_2 + c_1a_2 + c_2e_2) \tag{7}$$

holds, the system has a unique positive equilibrium point $A_4(x^*, y^*)$, where

$$\begin{aligned} x^* &= \frac{e_1c_2^2 - (a_2k_1 + c_2)(e_2k_2a_2 + c_1a_2 + c_2e_2)}{b_1c_2(c_2 + k_1a_2)}, \\ y^* &= \frac{a_2}{c_2}. \end{aligned} \tag{8}$$

Remark 2.1. Condition (7) is equivalent to the following condition

$$\frac{e_1}{1+k_1y^*} - e_2(1+k_2y^*) > c_1y^*. \tag{9}$$

Regarding the local stability of these equilibria, we have the following result:

Theorem 2.1. *Assume that (6) holds, the equilibrium points A_1, A_3 are unstable; if*

$$\frac{e_1}{1+k_1y^*} - e_2(1+k_2y^*) < c_1y^* \tag{10}$$

holds, the equilibrium point A_2 is locally asymptotically stable; the positive equilibrium point A_4 is locally asymptotically stable if (9) holds.

Proof. The variational matrix of the system represented by equation (4) evaluated at the point (x, y) is given by

$$J(x,y) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & -2c_2y + a_2 \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} A_{11} &= \frac{e_1}{k_1y+1} - e_2(k_2y+1) - 2b_1x - c_1y, \\ A_{12} &= x\Big(-\frac{e_1k_1}{(k_1y+1)^2} - e_2k_2 - c_1\Big). \end{aligned}$$

The characteristic equation of the variational matrix is

$$\lambda^2 - tr(J)\lambda + det(J) = 0. \tag{12}$$

Obviously, if $tr(J) < 0$ and $det(J) > 0$ hold true, the equation mentioned above has two roots with negative real parts. As a result, the characteristic roots of equation (4) also possess negative real parts. Consequently, the associated equilibrium point is locally asymptotically stable.

(1) For the equilibrium point $A_1(0,0)$, there is

$$\begin{aligned} tr\Big(J(0,0)\Big) &= e_1 - e_2 + a_2, \\ det\Big(J(0,0)\Big) &= (e_1 - e_2)a_2. \end{aligned}$$

Then $tr(J(0,0)) > 0, det(J(0,0)) > 0$ if (6) holds, so $A_1(0,0)$ is unstable;

(2) The Jacobian matrix of the system (4) at the equilibrium point $A_3(\frac{e_1-e_2}{b_1}, 0)$ is

$$J((\frac{e_1-e_2}{b_1}, 0)) = \begin{pmatrix} -e_1 + e_2 & \frac{\Delta_1}{b_1} \\ 0 & a_2 \end{pmatrix}, \tag{13}$$

where

$$\Delta_1 = (e_1 - e_2)(-e_1k_1 - e_2k_2 - c_1).$$

Assume that (6) holds, then the two eigenvalues of the matrix satisfy $\lambda_1 = -e_1 + e_2 < 0, \lambda_2 = a_2 > 0$. It can be seen that $A_3(\frac{e_1-e_2}{b_1}, 0)$ is unstable;

(3) The Jacobian matrix of the system (4) at the equilibrium point $A_2(0, \frac{a_2}{c_2})$ is

$$\begin{aligned} & J((0, \tfrac{a_2}{c_2})) \\ &= \begin{pmatrix} \Delta_2 & 0 \\ 0 & -a_2 \end{pmatrix} \\ &= \begin{pmatrix} \Delta_2^* & 0 \\ 0 & -a_2 \end{pmatrix}. \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Delta_2 &= \frac{e_1}{k_1\frac{a_2}{c_2}+1} - e_2(1+k_2\frac{a_2}{c_2}) - c_1\frac{a_2}{c_2}, \\ \Delta_2^* &= \frac{e_1}{k_1y^*+1} - e_2(1+k_2y^*) - c_1y^*. \end{aligned}$$

Under assumption (10), it is clear that

$$\begin{aligned}\lambda_1 &= \frac{e_1}{ky^*+1} - e_2 - c_1y^* < 0,\\ \lambda_2 &= -a_2 < 0.\end{aligned}$$

Correspondingly, $A_2(0, \frac{a_2}{c_2})$ is locally asymptotically stable. On the contrary, if (9) holds, then $\lambda_1 > 0$, thus $A_2(0, \frac{a_2}{c_2})$ is unstable;

(4) It should be noted that the positive equilibrium point $A_4(x^*, y^*)$ satisfies the following equation

$$\begin{aligned}\frac{e_1}{1+k_1y^*} - e_2(1+k_2y^*) - b_1x^* - c_1y^* &= 0,\\ a_2 - c_2y^* &= 0.\end{aligned} \quad (15)$$

With the help of (15), the Jacobian matrix of the system (4) at the positive equilibrium point $A_4(x^*, y^*)$ is

$$J((x^*, y^*)) = \begin{pmatrix} -b_1x^* & \Delta_3 \\ 0 & -c_2y^* \end{pmatrix}, \quad (16)$$

where

$$\Delta_3 = x^*\Big(-\frac{e_1k}{(k_1y^*+1)^2} - e_2k_2 - c_1\Big).$$

It is easy to see that

$$\begin{aligned} tr\Big(J(x^*, y^*)\Big) &= -b_1x^* - c_2y^* < 0,\\ det\Big(J(x^*, y^*)\Big) &= b_1x^*c_2y^* > 0.\end{aligned}$$

The local asymptotic stability of $A_4(x^*, y^*)$ can be observed.

Theorem 2.1 is proved.

III. Global attractivity

The research in the previous section shows that under appropriate conditions, the boundary equilibrium point $A_2(0, \frac{a_2}{c_2})$ and the positive equilibrium point $A_4(x^*, y^*)$ can be locally asymptotically stable. Now, an interesting question is whether or not they can be globally stable. This section further discusses this matter.

Theorem 3.1. Assuming that (10) holds, then $A_2(0, \frac{a_2}{c_2})$ is globally attractive.

Proof. Inequality (10) shows that for sufficiently small positive numbers $\varepsilon > 0$, the following inequality holds:

$$\frac{e_1}{1+k_1(y^*-\varepsilon)} - e_2(1+k_2(y^*-\varepsilon)) < c_1(y^*-\varepsilon). \quad (17)$$

Note that the second equation of the system (4)

$$\frac{dy}{dt} = y\Big(a_2 - c_2y\Big) \quad (18)$$

is the famous Logistic equation, and so

$$\lim_{t\to+\infty} y(t) = \frac{a_2}{c_2}. \quad (19)$$

For $\varepsilon > 0$ satisfying (17), from (19) we know that there is a sufficiently large T_1, so that when $t > T_1$, we have

$$y^* - \varepsilon = \frac{a_2}{c_2} - \varepsilon < y(t) < \frac{a_2}{c_2} + \varepsilon = y^* + \varepsilon. \quad (20)$$

When t exceeds T_1, it could be inferred from the first equation of (20) and (4) that

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{e_1}{1+k_1y} - e_2(1+k_2y) - b_1x - c_1y\Big)\\ &\le x\Big(\frac{e_1}{1+k_1(y^*-\varepsilon)} - e_2(1+k_2(y^*-\varepsilon))\\ &\quad -c_1(y^*-\varepsilon)\Big)\\ &\overset{def}{=} \Gamma x.\end{aligned} \quad (21)$$

Therefore, combined with (17), it can be seen that when $t \to +\infty$, there is

$$x(t) \le x(T_1)\exp\Big\{\Gamma(t-T_1)\Big\} \to 0.$$

That is to say, there are

$$\lim_{t\to+\infty} x(t) = 0. \quad (22)$$

(19) and (22) show that $A_2(0, \frac{a_2}{c_2})$ is globally attractive. Theorem 3.1 has been proved.

Theorem 3.2. Assuming (9) holds, then the unique positive equilibrium point $A_4(x^*, y^*)$ of the system is globally asymptotically stable.

Proof. If the condition (9) holds, it can be known from Theorem 2.1 that the boundary equilibrium points A_1, A_2, and A_3 are all unstable; the positive equilibrium point $A_4(x^*, y^*)$ is locally asymptotically stable.

If it can be established that the solution of the system (4) is bounded and limit cycles are absent, then, by the limit set theory of the planar system, it can be concluded that the solution of the system will approach the positive equilibrium point as time progresses. This implies that the positive equilibrium point $A_4(x^*, y^*)$ exhibits global asymptotic stability.

First, we prove that the solution of the system (4) with positive initial values is uniformly bounded. From the second equation of system (4), similar to the analysis of (18)-(19) in Theorem 3.1, we have

$$\lim_{t\to+\infty} y(t) = \frac{a_2}{c_2} \overset{def}{=} y^*. \quad (23)$$

It can be seen that for sufficiently small $\varepsilon > 0$, there exists $T_2 > 0$ such that when $t \ge T_2$, there is

$$y^* - \varepsilon < y(t) < y^* + \varepsilon. \quad (24)$$

Similar to the analysis of (21), from the first equations of the system (4) and (24), it can be seen that when $t > T_2$, there is

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{e_1}{1+k_1y} - e_2(1+k_2y) - b_1x - c_1y\Big)\\ &\le x\Big(\frac{e_1}{1+k(y^*-\varepsilon)} - e_2(1+k_2(y^*-\varepsilon)\\ &\quad -b_1x - c_1(y^*-\varepsilon)\Big),\end{aligned} \quad (25)$$

Now, let us consider the equation

$$\begin{aligned}\frac{du}{dt} &= u\Big(\frac{e_1}{1+k(y^*-\varepsilon)} - e_2(1+k_2(y^*-\varepsilon))\\ &\quad -b_1u - c_1(y^*-\varepsilon)\Big),\end{aligned} \quad (26)$$

From Lemma 3.1, we know that there are

$$\lim_{t\to+\infty} u(t) = W_1(\varepsilon), \tag{27}$$

where

$$W_1(\varepsilon) = \frac{\dfrac{e_1}{1+k(y^*-\varepsilon)} - e_2(1+k_2(y^*-\varepsilon)) - c_1(y^*-\varepsilon)}{b_1}.$$

From (26) and (27), with the help of differential inequality theory, we have

$$\limsup_{t\to+\infty} x(t) \le W_1(\varepsilon). \tag{28}$$

It can be seen that there is $T_3 > T_2$, when $t > T_3$,

$$x(t) < W_1(\varepsilon) + \varepsilon \overset{\text{def}}{=} \Gamma_1(\varepsilon). \tag{29}$$

Let

$$D = \{(x,y) \in R_+^2 : x < \Gamma_1(\varepsilon),\ \ y < y^* + \varepsilon\}.$$

Then, the solution of system (4) with positive initial value is eventually uniformly bounded in D. Now let us show that system (4) could not admits limit cycle in D. Consider the Dulac function $B(x,y) = x^{-1}y^{-1}$,

$$\frac{\partial(BF_1)}{\partial x} + \frac{\partial(BF_2)}{\partial y} \quad = \quad -\frac{b_1}{y} - \frac{c_2}{x} < 0,$$

where

$$F_1(x,y) = x\Big(\frac{e_1}{1+k_1y} - e_2(1+k_2y) - b_1x - c_1y\Big),$$

$$F_2(x,y) = y\Big(a_2 - c_2y\Big).$$

According to the Dulac criterion [43], the system (4) does not have closed orbits in D. The global asymptotic stability of $A_4(x^*, y^*)$ is evident, and the proof of Theorem 3.2 is ended.

Remark 3.1. If the system (4) has a unique positive equilibrium point, then the positive equilibrium point is globally asymptotically stable, according to Theorem 3.2.

Remark 3.2. According to Theorems 2.1, 3.1, and 3.2, the conditions that guarantee the local stability of the equilibrium point are adequate to guarantee its global asymptotic stability.

Remark 3.3. Noting that

$$\begin{aligned}
\frac{\partial x^*}{\partial k_1} &= -\frac{a_2e_1c_2}{b_1(k_1a_2+c_2)^2} < 0,\\
\frac{\partial y^*}{\partial k_1} &= 0,\\
\frac{\partial x^*}{\partial k_2} &= -\frac{a_2^2e_2k_1 + a_2c_2e_2}{b_1c_2(a_2k_1+c_2)} < 0,\\
\frac{\partial y^*}{\partial k_2} &= 0.
\end{aligned} \tag{30}$$

It can be seen that x^* is a monotonically decreasing function of the parameter k_1 and k_2, and the fear effect has no effect on the y species. The fear effect reduces the final density of the first species.

IV. Nonautonomous case

As we all know, the natural environment is constantly changes with time, so it is necessary to consider the non-autonomous case. However, as the coefficients change with time, the study becomes difficult.

In this section, we will study the non-autonomous case of the system

$$\begin{aligned}
\frac{dx}{dt} &= x\Big(\frac{e_1(t)}{1+k_1(t)y} - (1+k_2(t)y)e_2(t)\\
&\qquad -b_1(t)x - c_1(t)y\Big),\\
\frac{dy}{dt} &= y\Big(a_2(t) - c_2(t)y\Big).
\end{aligned} \tag{31}$$

Throughout this section, for a continuous and bounded function, we let $f^l = \inf_{t\in R} f(t)$ and $f^u = \sup_{t\in R} f(t)$.

In system (31), we always assume:

(H_1) $e_1(t), k_1(t), k_2(t), e_2(t), b_1(t), c_1(t), a_2(t)$, and $c_2(t)$ are all continuous and strictly positive functions that satisfy

$$\min\{e_1^l, k_1^l, k_2^l, e_2^l, b_1^l, c_1^l, a_2^l, c_2^l\} > 0,$$

$$\max\{e_1^u, k_1^u, k_2^u, e_2^u, b_1^u, c_1^u, a_2^u, c_2^u\} < +\infty.$$

Set

$$y^l \overset{def}{=} \frac{a_2^l}{c_2^u},\ \ y^u \overset{def}{=} \frac{a_2^u}{c_2^l}. \tag{32}$$

As far as nonautonomous biosystem is concerned, persistence, global attractivity, and extinction are the most important topics, which represent the survival or extinction of the species. Now we state and prove the main results of this section.

Theorem 4.1.
(1) Assuming

$$\frac{e_1^u}{1+k_1^ly^l} < e_2^l(1+k_2^ly^l) + c_1^ly^l \tag{33}$$

holds, then the first species $x(t)$ will be driven to extinction;
(2) Assuming

$$\frac{e_1^l}{1+k_1^uy^u} > e_2^u(1+k_2^uy^u) + c_1^uy^u \tag{34}$$

holds, then the system is permanent.

Proof. It follows from (33) and (34) that for enough small $\varepsilon > 0$, the following inequalities hold:

$$\frac{e_1^u}{1+k_1^l(y^l-\varepsilon)} < e_2^l(1+k_2^l(y^l-\varepsilon)) + c_1^l(y^l-\varepsilon), \tag{35}$$

$$\frac{e_1^l}{1+k_1^u(y^u+\varepsilon)} > e_2^u(1+k_2^u(y^u+\varepsilon)) + c_1^u(y^u+\varepsilon). \tag{36}$$

From the second equation of system (31) we have

$$y\Big(a_2^l - c_2^uy\Big) \le \frac{dy}{dt} \le y\Big(a_2^u - c_2^ly\Big), \tag{37}$$

thus, one has

$$y^l \overset{def}{=} \frac{a_2^l}{c_2^u} \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le \frac{a_2^u}{c_2^l} \overset{def}{=} y^u. \tag{38}$$

For $\varepsilon > 0$ enough small, which satisfies the inequality (35) and (36), there exists a $T > 0$ such that

$$y^l - \varepsilon < y(t) < y^u + \varepsilon, \quad t \geq T. \tag{39}$$

Now, from the first equation of (31) and (39), one has

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{e_1(t)}{1+k_1(t)y} - (1+k_2(t)y)e_2(t) \\ &\quad -b_1(t)x - c_1(t)y\Big) \\ &\leq x\Big(\frac{e_1^u}{1+k_1^l(y^l-\varepsilon)} - (1+k_2^l(y^l-\varepsilon))e_2^l \\ &\quad -b_1(t)x - c_1^l(y^l-\varepsilon)\Big). \end{aligned} \tag{40}$$

If condition (33) holds, then follows from (40) one has

$$x(t) \leq x(T)\exp\Big\{\Delta(\varepsilon)(t-T)\Big\} \to 0 \ \text{ as } \ t \to +\infty, \tag{41}$$

where

$$\begin{aligned} \Delta(\varepsilon) &= \frac{e_1^u}{1+k_1^l(y^l-\varepsilon)} - (1+k_2^l(y^l-\varepsilon))e_2^l \\ &\quad -c_1^l(y^l-\varepsilon). \end{aligned} \tag{42}$$

That is, if (33) holds, the first species $x(t)$ will be driven to extinction. This ends the proof of Theorem 4.1 (i).

Now assume that inequality (34) holds, then it immediately follows that

$$\frac{e_1^u}{1+k_1^l y^l} > e_2^l(1+k_2^l y^l) + c_2^l y^l, \tag{43}$$

so, for $\varepsilon > 0$ enough small, the following inequality holds

$$\frac{e_1^u}{1+k_1^l(y^l-\varepsilon)} > e_2^l(1+k_2^l(y^l-\varepsilon)) + c_1^l(y^l-\varepsilon). \tag{44}$$

Hence, it follows from (40) that

$$\limsup_{t\to+\infty} x(t) \leq \frac{\Delta(\varepsilon)}{b_1^u}, \tag{45}$$

where $\Delta(\varepsilon)$ is defined by (42). Setting $\varepsilon \to 0$ in (42) leads to

$$\limsup_{t\to+\infty} x(t) \leq \frac{\Delta}{b_1^u}, \tag{46}$$

where

$$\Delta = \frac{e_1^u}{1+k_1^l y^l} - (1+k_2^l y^l)e_2^l - c_1^l y^l. \tag{47}$$

Again, from the first equation of (31) and (39), one has

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{e_1(t)}{1+k_1(t)y} - (1+k_2(t)y)e_2(t) \\ &\quad -b_1(t)x - c_1(t)y\Big) \\ &\geq x\Big(\frac{e_1^l}{1+k_1^u(y^u+\varepsilon)} - (1+k_2^u(y^u+\varepsilon))e_2^u \\ &\quad -b_1^u x - c_1^u(y^u+\varepsilon)\Big). \end{aligned} \tag{48}$$

If condition (34) holds, then follows from (36) one has

$$\liminf_{t\to+\infty} x(t) \geq \frac{\Delta_1(\varepsilon)}{b_1^l}, \tag{49}$$

where

$$\begin{aligned} \Delta_1(\varepsilon) &= \frac{e_1^l}{1+k_1^u(y^u+\varepsilon)} - (1+k_2^u(y^u+\varepsilon))e_2^u \\ &\quad -c_1^u(y^u+\varepsilon). \end{aligned} \tag{50}$$

Setting $\varepsilon \to 0$ in (49) leads to

$$\liminf_{t\to+\infty} x(t) \geq \frac{\Delta_1}{b_1^l}, \tag{51}$$

where

$$\Delta_1 = \frac{e_1^l}{1+k_1^u y^u} - (1+k_2^u y^u)e_2^u - c_1^u y^u. \tag{52}$$

(38), (46), and (51) show that under the assumption (34) holds, the system is permanent. This ends the proof of Theorem 4.1 (ii).

The proof of Theorem 4.1 is ended.

Concerned with the global attractivity of the positive solutions of the system, we have the following result.

Theorem 4.2 *Let $(x^*(t), y^*(t))$ be a bounded positive solution of system (31). In addition to (34), assume further that the following inequality holds:*

$$c_2^l > \frac{e_1^u k_1^u}{(1+k_1^l y^l)^2} + e_2^u k_2^u + c_1^u, \tag{53}$$

where the variables y^l are specified by equations (38). Then $(x^*(t), y^*(t))$ exhibits global asymptotic stability.

Proof. The condition expressed in inequality (53) indicates that, for a sufficiently small positive constant $\varepsilon > 0$, it is possible to assume, without loss of generality, that $\varepsilon < \frac{1}{2}y^l$. Under this assumption, the following inequality is valid.

$$c_2^l > \frac{e_1^u k_1^u}{(1+k_1^l(y^l-\varepsilon))^2} + e_2^u k_2^u + c_1^u. \tag{54}$$

Consider the positive solution $(x(t), u(t))$ of equation (31), it may be deduced from condition (34) and Theorem 4.1 that, given any positive value of ε, there exists a positive value of T such that for all values of t greater than or equal to T,

$$y^l - \varepsilon < y(t), y^*(t) < y^u + \varepsilon. \tag{55}$$

For $t \geq T$, let us consider a Lyapunov function that is defined by

$$\begin{aligned} V(t) &= |\ln\{x(t)\} - \ln\{x^*(t)\}| \\ &\quad +|\ln\{y(t)\} - \ln\{y^*(t)\}|. \end{aligned} \tag{56}$$

We are now estimating and computing the upper right derivative of $V(t)$ along the positive solutions of the system

(31) for $t > T$. Applying (54) yields the following results:

$$
\begin{aligned}
& D^+V(t) \\
= \;& sgn(x(t)-x^*(t))\Big[-\frac{e_1(t)}{1+k_1(t)y^*(t)} \\
& +e_2(t)(1+k_2(t)y^*(t)) - e_2(t)(1+k_2(t)y(t)) \\
& +\frac{e_1(t)}{1+k_1(t)y(t)} + b_1(t)x^*(t) - b_1(t)x(t) \\
& +c_1(t)y^*(t) - c_1(t)y(t)\Big] \\
& +sgn(y(t)-y^*(t))\Big[-c_2(t)y^*(t)+c_2(t)y(t)\Big] \\
= \;& sgn(x(t)-x^*(t))\Big[\frac{e_1(t)k_1(t)(y(t)-y^*(t))}{(1+k_1(t)y^*(t))(1+k_1(t)y(t))} \\
& -e_2(t)k_2(t)(y(t)-y^*(t)) - b_1(t)(x(t)-x^*(t)) \\
& -c_1(t)(y(t)-y^*(t))\Big] \\
& +sgn(y(t)-y^*(t))\Big[-c_2(t)y^*(t)+c_2(t)y(t)\Big] \\
\leq \;& -\Gamma_1|x(t)-x^*(t)| - \Gamma_2|u(t)-u^*(t)|,
\end{aligned} \tag{57}
$$

where

$$
\begin{aligned}
\Gamma_1 &= b_1^l > 0, \\
\Gamma_2^\varepsilon &= c_2^l - \frac{e_1^u k_1^u}{(1+k_1^l(y^l-\varepsilon))^2} - e_2^u k_2^u - c_1^u > 0.
\end{aligned} \tag{58}
$$

For $t \geq T$, one thus has

$$
D^+V(t) \leq -\mu\Big(|x(t)-x^*(t)| + |y(t)-y^*(t)|\Big), \tag{59}
$$

where $\mu = \min\{\Gamma_1, \Gamma_2^\varepsilon\}$. Performing integration on both sides of equation (59) with respect to the variable t across the interval from T to t yields

$$
\begin{aligned}
V(t) \quad & +\mu\int_T^t \Big(|x(s)-x^*(s)| + |y(s)-y^*(s)|\Big)ds \\
& \leq V(T) < +\infty, \;\; t \geq T.
\end{aligned}
$$

Then, for all $t \geq T$,

$$
\int_T^t \Big(|x(s)-x^*(s)|+|y(s)-y^*(s)|\Big)ds \leq \mu^{-1}V(T) < +\infty,
$$

and hence,

$$
|x(t)-x^*(t)| + |y(t)-y^*(t)| \in L^1([T,+\infty)).
$$

The fact that $x^*(t)$ and $y^*(t)$ are bounded, and that $x(t)$ and $y(t)$ are ultimately bounded, implies that the derivatives of $x(t)$, $x^*(t)$, $y(t)$, and $y^*(t)$ are all bounded for $t \geq T$, as indicated by the equations that govern their behavior. Consequently, it may be inferred that $|x(t)-x^*(t)|+|y(t)-y^*(t)|$ is uniformly continuous on $[T,+\infty)$. Thus, by Barbălat's Lemma[38], we have

$$
\lim_{t\to+\infty}\Big(|x(t)-x^*(t)| + |y(t)-y^*(t)|\Big) = 0.
$$

The proof is completed.

Example 5.3. Now let's consider the following non-autonomous case. Take $e_2 = a_1 = b_1 = 1, c_1 = \frac{1}{2}, a_2 = 3+\cos(t), c_2 = 3-\sin(t), e_1 = 5-\sin(t), k_1 = \frac{1}{10}, k_2 = \frac{1}{2}$. Then, by simple computation, we have $y^l = \frac{1}{2}, y^u = 2$, thus

$$
\frac{e_1^l}{1+k_1^u y^u} = \frac{10}{3} > 3 = e_2^u(1+k_2^u y^u) + c_1^u y^u \tag{60}
$$

and

$$
c_2^l = 2 > \frac{0.6}{(1.05)^2} + 1 = \frac{e_1^u k_1^u}{(1+k_1^l y^l)^2} + e_2^u k_2^u + c_1^u. \tag{61}
$$

From Theorem 4.2, we know that the system is permanent and the positive solution of the system is globally asymptotically stable. Figures 6 verifies this fact.

V. Numeric simulations

Now let us consider the following three examples.

Example 5.1. Take $e_2 = b_1 = c_1 = a_2 = c_2 = 1$.
(1) Take $e_1 = 2, k_1 = 0.5$, and $k_2 = 1$. At this time, the calculation shows that $y^* = 1$,

$$
\frac{e_1}{1+k_1y^*} - e_2(1+k_2y^*) = -\frac{2}{3} < 1 = c_1y^*. \tag{62}
$$

From Theorem 3.1, we know that $A_2(0,1)$ is globally attractive. Figures 1 and 2 verify this fact;
(2) Take $e_1 = 3, k = 0.1, k_2 = 1$. At this time, the calculation shows that $y^* = 1$,

$$
\frac{e_1}{1+k_1y^*} - e_2(1+k_2y^*) = \frac{18}{11} > 1 = c_1y^*. \tag{63}
$$

From Theorem 3.2, we know that $A_4(0.6363636364, 1)$ is globally asymptotically stable. Figure 3 verifies this fact.

Example 5.2. Take $e_1 = 4, e_2 = 0.1, b_1 = c_1 = a_2 = c_2 = 1$. In this instance, $x*$ is a solution to the equation

$$
\frac{4}{1+k_1} - 1.1 - 0.1k_2 - x = 0. \tag{64}
$$

(1) Take $k_2 = 1$. According to the data presented in Figure 4, the variable x^* exhibits a consistent pattern of decreasing values as the parameter k_1 increases. Furthermore, as k_1 reaches sufficiently large values, the variable x^* converges towards zero, which means that the first population eventually tends to extinction.
(2) Take $k_1 = 1$. According to the data presented in Figure 5, the variable x^* exhibits a consistent pattern of decreasing values as the parameter k_2 increases. Additionally, as k_2 reaches sufficiently large values, it can be inferred that x^* approaches a limit of zero, which means that the first population eventually tends to extinction.

Example 5.3. Now, let us consider the following non-autonomous case. Take $e_2 = a_1 = b_1 = 1, c_1 = \frac{1}{2}, a_2 = 3+\cos(t), c_2 = 3-\sin(t), e_1 = 5-\sin(t), k_1 = \frac{1}{10}, k_2 = \frac{1}{2}$. Then, by simple computation, we have $y^l = \frac{1}{2}, y^u = 2$, thus

$$
\frac{e_1^l}{1+k_1^u y^u} = \frac{10}{3} > 3 = e_2^u(1+k_2^u y^u) + c_1^u y^u \tag{65}
$$

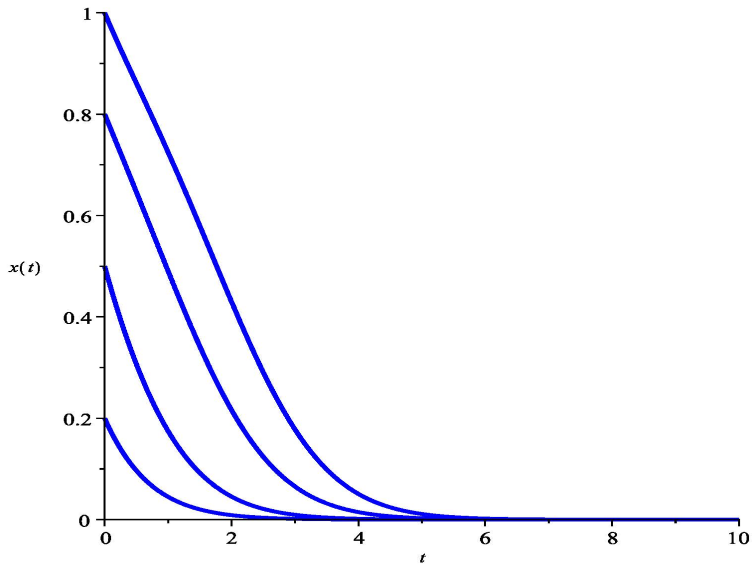

Fig. 1. Numeric simulations of $x(t)$ in Example 5.1, Case 1, the initial condition $(x(0), y(0)) = (1, 0.1)$, $(0.8, 0.2)$, $(0.2, 0.8)$, and $(0.5, 0.5)$, respectively.

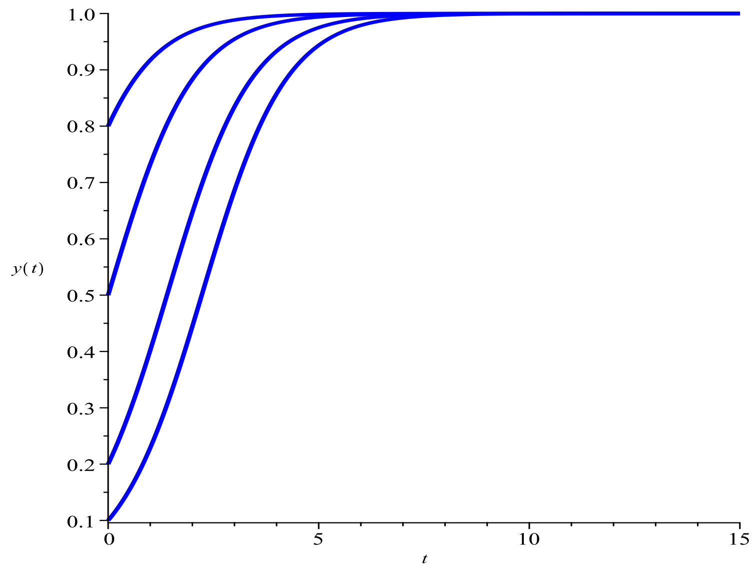

Fig. 2. Numeric simulations of $y(t)$ in Example 5.1, Case 1, the initial condition $(x(0), y(0)) = (1, 0.1)$, $(0.8, 0.2)$, $(0.2, 0.8)$, and $(0.5, 0.5)$, respectively.

and

$$c_2^l = 2 > \frac{0.6}{(1.05)^2} + 1 = \frac{e_1^u k_1^u}{(1 + k_1^l y^l)^2} + e_2^u k_2^u + c_1^u. \quad (66)$$

From Theorem 4.2, we know that the system is permanent, and the positive solution of the system is globally asymptotically stable. Figure 6 verifies this fact.

VI. Discussion

Xi, Griffin, and Sun[1] pointed out that grassland caterpillars and grasshoppers are two main herbivorous insects. Among them, the grasshoppers' natural jumping seriously reduces the feeding time of grassland caterpillars, ultimately leading to a decline in the number of female caterpillars laying eggs. The study of fear in this amensalism relationship has yet to be reported. Assuming that amensalism can reduce the birth rate and increase the death rate of the affected species, this paper proposes an amensalism model with fear effects for the first time and discusses its dynamic behaviors. This study demonstrates that the fear effect played a significant role in driving the extinction of the first species.

Note that our Theorems 3.1 and 3.2 reduce to Theorem A when $k_1 = k_2 = 0$. That is, we generalize the main results of Zhu and Chen[12] to more general cases. Our study shows that the fear effect is an important ecological factor influencing the dynamic behavior of amensalism systems. However, this is only a theoretical study, and it is necessary to combine the observations of field experiments to further

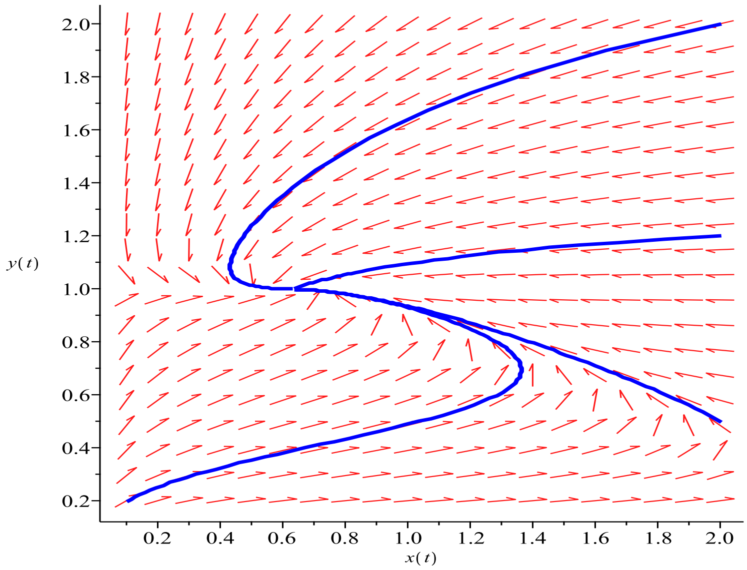

Fig. 3. Phase trajectories of $(x(t), y(t))$ in Example 5.1, Case 2, the initial condition $(x(0), y(0)) = (2, 2)$, $(2, 1.2)$, $(0.1, 0.2)$, and $(2, 0.5)$, respectively.

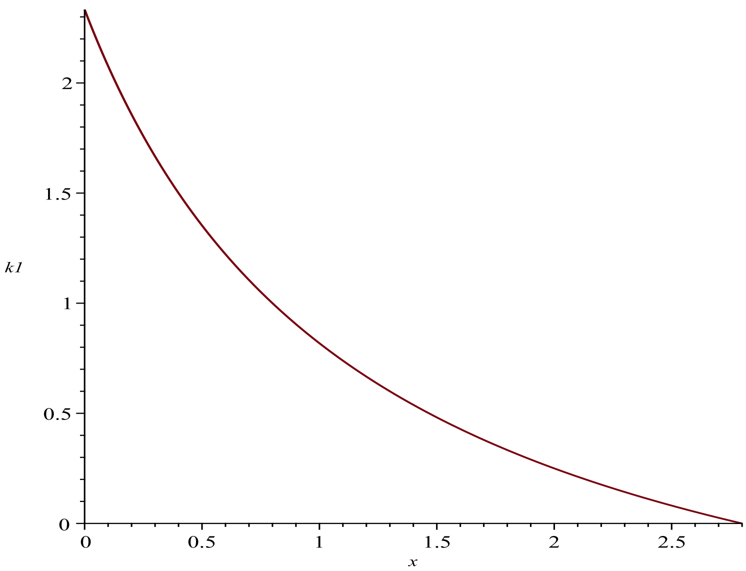

Fig. 4. Relationship of x^* and k_1.

propose a more reasonable model and carry out more targeted research.

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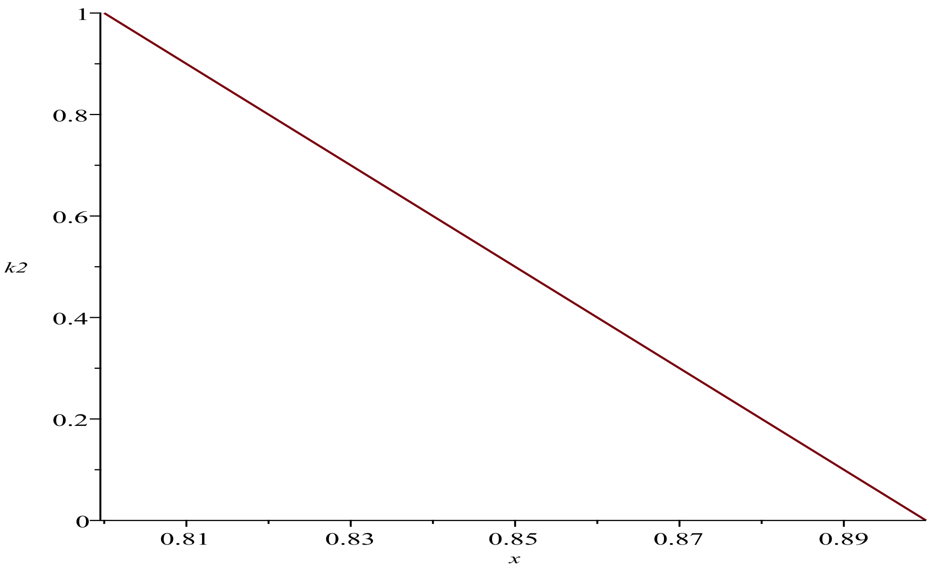

Fig. 5. Relationship of x^* and k_2.

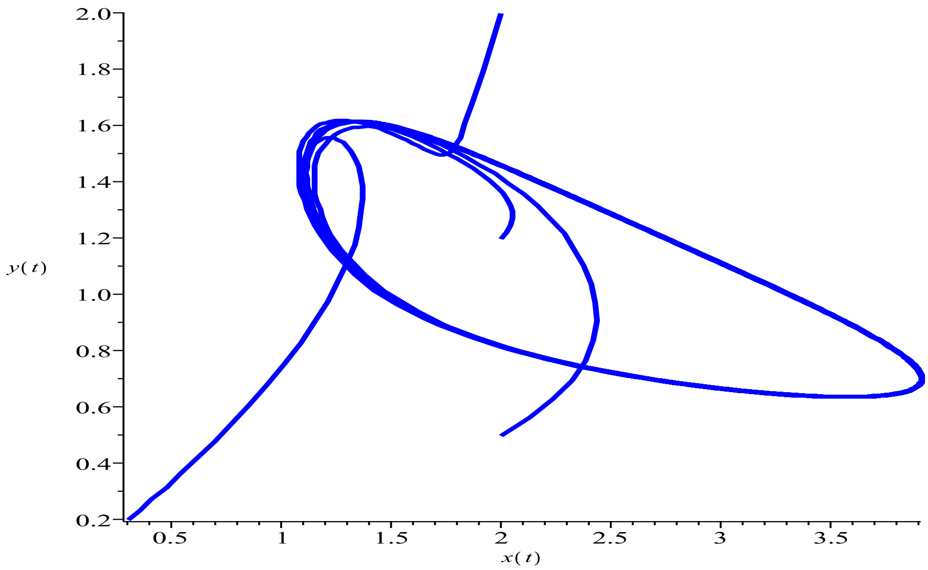

Fig. 6. Phase trajectories of $(x(t), y(t))$ in Example 5.3, the initial condition $(x(0), y(0)) = (2, 2)$, $(2, 1.2)$, $(0.3, 0.2)$, and $(2, 0.5)$, respectively.

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