Further Characterizations of Secondary Generalized Inverse

Divya Shenoy Purushothama* and Pruthvi K K

Abstract—**In this article, certain equivalent conditions for the existence of the secondary generalized inverse are obtained. Several new characterizations of secondary generalized inverse are presented. Also, the representation of $\{1,3^\theta\}$, $\{1,4^\theta\}$, $\{1,2,3^\theta\}$ and $\{1,2,4^\theta\}$ are given here.**

Index Terms—**Moore-Penrose inverse, secondary generalized inverse, secondary transpose**

I. Introduction

THE concept of secondary transpose of a matrix is introduced by Anna Lee [4] by reflecting the entries through the secondary diagonal. The secondary transpose A^s (denoted as A^θ in the case of conjugate secondary transpose) is related to the classical transpose by the relation $A^\theta = VA^*V$ where V is a unit secondary diagonal matrix, i.e., the matrix V has entries '1' on its secondary diagonal and all the remaining entries are zeros. Using the involution operator 'θ', the secondary generalized inverse is defined by Vijayakumar [9] and later modified by Savitha et al. [5]. Even though their definition is similar to the well-known Moore-Penrose inverse and Minkowski inverse, it can be noted that the existence of secondary generalized inverse is not always assured. For different characterizations of Minkowski inverse, refer [2], [3], [13]. For more ideas related to secondary transpose one can refer [6], [7], [8], [10]. A determinantal representation and some characterizations of secondary generalized inverse are given in [5]. In this article, we further characterize the secondary generalized inverse in line with the characterization of the Minkowski inverse [2].

Here are some preliminary results and notations.

II. Preliminaries

Throughout this paper, $A \in \mathbb{C}^{m\times n}$ denotes that A is an $m\times n$ matrix over a complex field. $\mathscr{C}(A)$ and $\mathscr{N}(A)$ are the the column space and the null space of A, respectively.

Definition 1. *[5] An $n\times m$ matrix G is called the secondary generalized inverse of A if it satisfies the following conditions: (1) $AGA = A$ (2) $GAG = G$ (3)$(AG)^\theta = AG$ (4)$(GA)^\theta = GA$*

The matrix G is denoted by $A^{\dagger_\theta}$, and is unique whenever it exists. For any $A \in \mathbb{C}^{m\times n}$, let $A\{1,2,3^\theta\}$ denotes the set of matrices satisfying $(1),(2),(3)$ and $A\{1,2,4^\theta\}$ is the set of matrices satisfying $(1),(2),(4)$. $P_{L,M}$ is the projector on L along M, where L and M are two subspaces of $\mathbb{C}^n$ such that $L\oplus M = \mathbb{C}^n$.

Lemma 1. *[5] Given an $m\times n$ matrix A, the following statements are equivalent.*

1) *A has s-cancellation property.*
2) *$rank(AA^\theta) = rank(A^\theta A) = rank(A)$*
3) *$A^{\dagger_\theta}$ exists and*

$$A^{\dagger_\theta} = A^\theta(AA^\theta)^- A(A^\theta A)^- A^\theta$$

where $(AA^\theta)^-$ and $(A^\theta A)^-$ are arbitrary generalized inverses of AA^θ and $A^\theta A$ respectively.

The following theorem depicts the application of generalized inverses in solving matrix equations.

Theorem 1. *[12] Let $A \in \mathbb{C}^{m\times n}$,$B \in \mathbb{C}^{p\times q}$ and $D \in \mathbb{C}^{m\times q}$. Then, there is a solution $X \in \mathbb{C}^{n\times p}$ to the matrix equation $AXB = D$ if and only if, for some $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$, $AA^{(1)}DB^{(1)}B = D$; In this case, the general soltion is*

$$X = A^{(1)}DB^{(1)} + (I_n - A^{(1)}A)Y + Z(I_p - BB^{(1)})$$

where $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$ are fixed but arbitrary; and $Y \in \mathbb{C}^{n\times p}$ and $Z \in \mathbb{C}^{n\times p}$ are arbitrary.

III. Results

A determinantal formula for secondary generalized inverse is given by Savitha et al. [5]. Further characterizations for secondary generalized inverse in terms of rank factorizations, rank conditions and projectors are obtained here.

The following lemma gives a formula for calculating the secondary generalized inverse (s-g inverse) of a secondary diagonal matrix (s – diagonal matrix).

Lemma 2. *If D is a secondary diagonal matrix, such that $D = s - diag(d_1, d_2, ..., d_n)$, then $D^{\dagger_\theta} = s - diag(d_n^{\dagger_\theta}, d_{n-1}^{\dagger_\theta}, ..., d_1^{\dagger_\theta})$, whenever $D^{\dagger_\theta}$ exists.*

Proof: The result can be directly verified by the definition of secondary generalized inverse. Also, note that

$$d_i^{\dagger_\theta} = \begin{cases} \dfrac{1}{d_i} & \text{if } d_i \neq 0 \\ 0 & \text{if } d_i = 0 \end{cases} \qquad for \quad i = 1,2,...n.$$

∎

The following example demonstrates Lemma 2.

Example 1. *Consider a secondary diagonal matrix* $D = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, *the s-g inverse is* $D^{\dagger_\theta} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$

The Moore-Penrose inverse exists for all matrices, and Greville [1] gives a formula for the same using rank factorization. However, the secondary generalized inverse does

Manuscript received March 30, 2024; revised October 5, 2024.

Divya Purushothama Shenoy is an Assistant Professor at the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India - 576104. e-mail: divya.shenoy@manipal.edu

Pruthvi K K is a post graduate student at the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India - 576104. e-mail: pruthvikanuvalli@gmail.com

not exist for all matrices. The following example depicts the importance of imposing certain restrictions on the factors of rank factorization of a matrix to obtain the secondary generalized inverse.

Example 2. *Let* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. *Clearly, the rank of the matrix is two, and the rank factorization is given by*
$A = FG = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Here, $F^\theta A G^\theta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and hence we can not proceed further.*

This leads to imposing certain restrictions on F and G to obtain a formula for secondary generalized inverse, whenever exists.

Theorem 2. *Let* $A \in \mathbb{C}^{m\times n}$ *matrix with rank* r *be written as the product of two matrices* F *and* G, *i.e.,*

$$A = FG \tag{1}$$

where $F \in \mathbb{C}^{m\times r}$ *and* $G \in \mathbb{C}^{r\times n}$ *have full column rank and full row rank, respectively. Then secondary generalized inverse of* A,

$$A^{\dagger_\theta} = G^\theta (F^\theta A G^\theta)^{-1} F^\theta. \tag{2}$$

where F^θ *and* G^θ *are the secondary conjugate transpose of matrices* F *and* G *satisfying the conditions*
$rank(F) = rank(F^\theta) = rank(FF^\theta) = rank(F^\theta F)$ *and*
$rank(G) = rank(G^\theta) = rank(GG^\theta) = rank(G^\theta G)$
respectively.

Proof: To show that $F^\theta A G^\theta$ is nonsingular, by (1) we have

$$F^\theta A G^\theta = (F^\theta F)(GG^\theta) \tag{3}$$

And both the factors of RHS of (3) are r × r matrices. Thus, $F^\theta A G^\theta$ is the product of two nonsingular matrices
$\therefore F^\theta A G^\theta$ is nonsingular and hence

$$(F^\theta A G^\theta)^{-1} = (G^\theta G)^{-1} (F^\theta F)^{-1}.$$

Now, it can be easily verified that

$$X = G^\theta (G^\theta G)^{-1} (F^\theta F)^{-1} F^\theta$$

satisfies all conditions given in definition 1. ■

This is demonstrated using example 3.

Example 3. *Let* $A = \begin{bmatrix} 3 & 6 & 13 \\ 2 & 4 & 9 \\ 1 & 2 & 3 \end{bmatrix}$. *Here, rank of the matrix is 2 and the rank factorization is*
$A = \begin{bmatrix} 3 & 6 & 13 \\ 2 & 4 & 9 \\ 1 & 2 & 3 \end{bmatrix} = FG = \begin{bmatrix} 3 & 13 \\ 2 & 9 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now,

$$\begin{aligned} A^{\dagger_\theta} &= G^\theta (F^\theta A G^\theta)^{-1} F^\theta \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & -319/10 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 9 & 13 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -79/10 & 41/5 & 83/10 \\ 2 & -2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \end{aligned}$$

Also, it can be noted that the Moore Penrose inverse $A^\dagger = \begin{bmatrix} -3/130 & -11/65 & 79/130 \\ -3/65 & -22/65 & 79/65 \\ 1/13 & 3/13 & -9/13 \end{bmatrix}$ of the matrix A is different from the secondary generalized inverse $A^{\dagger_\theta}$ of A.

Lemma 3. *Consider two matrices* $A \in \mathbb{C}^{m\times p}$ *and* $B \in \mathbb{C}^{q\times m}$. *Then*

$$(P_{\mathscr{C}(A),\mathscr{N}(B)})^\theta = P_{\mathscr{C}(B^\theta),\mathscr{N}(A^\theta)}$$

Proof: Let $R = (P_{\mathscr{C}(A),\mathscr{N}(B)})^\theta = VP_{(\mathscr{N}(B))^\perp,(\mathscr{C}(A))^\perp}V$. Now $R^2 = R$, and hence

$$\mathscr{C}(R) = V(\mathscr{N}(B))^\perp = \mathscr{C}(VB^*) = \mathscr{C}(B^\theta),$$

Now,
$(\mathscr{N}(R))^\perp = (\mathscr{N}(P_{(\mathscr{N}(B)^\perp,\mathscr{C}(B)^\perp)}V))^\perp = \mathscr{C}(V(P_{\mathscr{C}(A),\mathscr{N}(B)}) = \mathscr{C}(VA) = \mathscr{C}(A^\theta)^*)$, implies $\mathscr{N}(R) = (\mathscr{C}(A^\theta)^*)^\perp = \mathscr{N}(A^\theta)$. Hence, $R = P_{\mathscr{C}(B^\theta),\mathscr{N}(A^\theta)}$. ■

Characterizations for $A\{1,2,3^\theta\}$, and $A\{1,2,4^\theta\}$ using rank factorization are given below:

Theorem 3. *Consider* $A \in \mathbb{C}^{m\times n}$ *and* $G \in \mathbb{C}^{n\times m}$. *Then, the following statements are equivalent.*

1) $G \in A\{1,3^\theta\}$;
2) $A^\theta A G = A^\theta$;
3) $AG = P_{\mathscr{C}(A),\mathscr{N}(A^\theta)}$

In this case, $A\{1,3^\theta\} = \{A^{(1,3^\theta)} + (I_n - A^{(1,3^\theta)}A)Y | Y \in \mathbb{C}^{n\times m}\}$ *where* $A^{1,3^\theta} \in A\{1,3^\theta\}$ *is fixed, but arbitrary.*

Proof: (1) $\Longrightarrow$ (2).Since, $G \in A\{1,3^\theta\}$,
$A^\theta A G = A^\theta (AG)^\theta = (AGA)^\theta = A^\theta$.
(2) $\Longrightarrow$ (3) Since $(AG)^\theta A = A$, from condition (2) we have, $AG = (AG)^\theta AG$, and this results in $(AG)^\theta = AG$. Thus, $AG = (AG)^\theta AG = (AG)^2$, ie., AG is a projector.
Now, by $(AG)^\theta = AG$, we have $AGA = A$, which, together with $A^\theta AG = A^\theta$, shows that $\mathscr{C}(AG) = \mathscr{C}(A)$ and $\mathscr{N}(AG) = \mathscr{N}(A^\theta)$. Hence $AG = P_{\mathscr{C}(A),\mathscr{N}(A^\theta)}$.
(3) $\Longrightarrow$ (1). Clearly, $AGA = P_{\mathscr{C}(A),\mathscr{N}(A^\theta)}A = A$. Applying Lemma 3 to $AG = P_{\mathscr{C}(A),\mathscr{N}(A^\theta)}$, we get

$$(AG)^\theta = P_{\mathscr{C}(A),\mathscr{N}(A^\theta)} = AG.$$

Also, for a fixed $A^{(1,3^\theta)}$, we have

$$A\{1,3^\theta\} = \{X \in \mathbb{C}^{n\times m} |\ AZ = AA^{(1,3^\theta)}\}.$$

Now by Lemma 3 we get

$$A\{1,3^\theta\} = \{A^{(1,3^\theta)} + (I_n - A^{(1,3^\theta)}A)Y | Y \in \mathbb{C}^{n\times m}\}$$

directly. ■

Theorem 4. *Consider an $m \times n$ matrix A with $rank(A) = rank(A^\theta A) = r$. Let $A = PQ$ be the full rank factorization of A such that $P \in \mathbb{C}_r^{m\times r}$ and $Q \in \mathbb{C}_r^{r\times n}$. Then*

$$A\{1,2,3^\theta\} = \{Q_R^{-1}(P^\theta P)^{-1}P^\theta | Q_R^{-1} \text{is an arbitrary right inverse of } Q\}$$

Proof: $Q_R^{-1}(P^\theta P)^{-1}P^\theta \in A\{1,2,3^\theta\}$ can be verified easily. To prove the converse, let $X \in A\{1,2,3^\theta\}$. Since $X \in A\{1,2\}$, we have $X = Q_R^{-1}P_L^{-1}$, for some $P_L^{-1} \in \mathbb{C}_r^{r\times m}$ and $Q_R^{-1} \in \mathbb{C}_r^{n\times r}$. Also, as $X \in A\{3^\theta\}$ we have that $(AX)^\theta = AX$ if and only if $(PP_L^{-1})^\theta = PP_L^{-1}$ iff $P_L^{-1} = (P^\theta P)^{-1}P^\theta$.
This implies that every $X \in A\{1,2,3^\theta\}$ must be of the form $Q_R^{-1}(P^\theta P)^{-1}P^\theta$. Hence the proof and the following example. ■

Example 4. *Let the matrix A be as in Example 3.* $A = \begin{bmatrix}3 & 6 & 13\\ 2 & 4 & 9\\ 1 & 2 & 3\end{bmatrix} = PQ = \begin{bmatrix}3 & 13\\ 2 & 9\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$.
Choose the arbitrary right inverse Q_R^{-1} of Q be

$$Q_R^{-1} = \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}.$$

Hence,

$$A\{1,2,3^\theta\} = Q_R^{-1}(P^\theta P)^{-1}P^\theta = \begin{bmatrix}-39/10 & 21/5 & 43/10\\ 0 & 0 & 0\\ 1 & -1 & -1\end{bmatrix}.$$

Theorem 5. *Consider $A \in \mathbb{C}^{m\times n}$ and $G \in \mathbb{C}^{n\times m}$. Then, the following statements are equivalent.*

1) $G \in A\{1,4^\theta\}$;
2) $GAA^\theta = A^\theta$;
3) $GA = P_{\mathscr{C}(A^\theta),\mathscr{N}(A)}$

In this case, $A\{1,4^\theta\} = \{A^{(1,4^\theta)} + Z(I_m - A^{(1,4^\theta)}A) | Z \in \mathbb{C}^{n\times m}\}$ where $A^{(1,4^\theta)} \in A\{1,4^\theta\}$ is fixed, but arbitrary.

Proof of this theorem follows similar lines to theorem 3.

Theorem 6. *Consider an $m \times n$ matrix A with $rank(A) = rank(A^\theta A) = r$. Let $A = PQ$ be the full rank factorization of A such that $P \in \mathbb{C}_r^{m\times r}$ and $Q \in \mathbb{C}_r^{r\times n}$. Then*

$$A\{1,2,4^\theta\} = \{Q^\theta(QQ^\theta)^{-1}P_L^{-1} | P_L^{-1} \text{is an arbitrary right inverse of } Q\}$$

This theorem is illustrated in the following example.

Example 5. *For* $A = \begin{bmatrix}3 & 6 & 13\\ 2 & 4 & 9\\ 1 & 2 & 3\end{bmatrix} = PQ = \begin{bmatrix}3 & 13\\ 2 & 9\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$,
the arbitrary left inverse of P_L^{-1} of P is $\begin{bmatrix}0 & -1 & 3\\ 0 & 1/3 & -2/3\end{bmatrix}$.
Hence,

$$A\{1,2,4^\theta\} = Q^\theta(QQ^\theta)^{-1}P_L^{-1} = \begin{bmatrix}0 & -7/3 & 17/3\\ 0 & 2/3 & -4/3\\ 0 & 1/3 & -2/3\end{bmatrix}.$$

Theorem 7. *For any finite $m \times n$ matrix A with $rank(A) = rank(A^\theta) = rank(AA^\theta) = rank(A^\theta A)$, then the s-g inverse $A^{\dagger\theta} = ZAY$ where $Y = (A^\theta A)^{(1)}A^\theta$ and $Z = A^\theta(AA^\theta)^{(1)}$ such that $Y \in A\{1,2,3^\theta\}$ and $Z \in A\{1,2,4^\theta\}$ respectively.*

Proof: Since

$$rank(A) = rank(A^\theta) = rank(AA^\theta) = rank(A^\theta A),$$

we have $\mathscr{C}(A^\theta A) = \mathscr{C}(A) = \mathscr{C}(A^\theta)$ for a finite matrix A. Then,

$$A^\theta = A^\theta AU \tag{4}$$

for some matrix U. On taking secondary conjugate transpose, we get

$$A = U^\theta A^\theta A. \tag{5}$$

As a consequence, $AYA = U^\theta A^\theta A(A^\theta A)^{(1)}A^\theta A = A$, Thus Y satisfies condition (1) of s-g inverse.
But rank(Y) $\geq$ rank(A) and, by the definition of Y,

$$rank(Y) \leq rank(A^\theta) = rank(A).$$

Therefore,

$$\text{rank}(Y) = \text{rank}(A),$$

and hence $Y \in$ A{1,2}. Using equations (4) and (5)

$$\begin{aligned}AY &= A(A^\theta A)^{(1)}A^\theta\\ (AY)^\theta &= (A(A^\theta A)^{(1)}A^\theta)^\theta\\ &= (A(A^\theta A)^{(1)}A^\theta AU)^\theta\\ &= U^\theta A^\theta A(A^\theta A)^{(1)}A^\theta\\ &= A(A^\theta A)^{(1)}A^\theta\end{aligned}$$

This implies $AY = (AY)^\theta$. Thus, the condition (3) of definition 1 is established.
Similarly, we can have

$$AZA = U^\theta A^\theta AA^\theta(AA^\theta)^{(1)}A^\theta A = U^\theta A^\theta A = A$$

Thus $Z \in A\{1\}$.
However, rank(Z) $\geq$ rank(A) and by the definition of Z,

$$rank(Z) \leq rank(A^\theta) = rank(A).$$

Therefore $rank(Z) = rank(A)$ and hence $Z \in A\{1,2\}$.
Now, using the equations (4) and (5)

$$\begin{aligned}ZA &= A^\theta(AA^\theta)^{(1)}A = A^\theta AU(AA^\theta)^{(1)}U^\theta A^\theta A\\ (ZA)^\theta &= (A^\theta AU(AA^\theta)^{(1)}U^\theta A^\theta A)^\theta\\ &= A^\theta AU(AA^\theta)^{(1)}U^\theta A^\theta A = ZA\end{aligned}$$

Now consider,

$$A^{\dagger\theta} = ZAY \tag{6}$$

Let X denotes RHS of (6).
Let $X = ZAY$. Since Y, Z satisfies condition (1) of s-g inverse, it can be easily verified that $X \in A\{1,2\}$. Moreover,(5) gives $AX = AY$, $XA = ZA$.
But by the definition of Y and Z,
$ZA = (ZA)^\theta$ and $AY = (AY)^\theta$.
Thus X satisfies all conditions given in definition 1. Therefore $A^{\dagger\theta} = X$. ■

Remark 1. *It can be easily verified that whenever $A^{\dagger\theta} \in \mathbb{C}^{m\times n}$ exists, the secondary generalized inverse is $A^{\dagger\theta} = A^{(1,4^\theta)}AA^{(1,3^\theta)}$.*

From Example 3, 4, 5, and 6 it can be verified that

$$A^{\dagger\theta} = A^{(1,4^\theta)}AA^{(1,3^\theta)} =$$

$$\begin{bmatrix} 0 & -7/3 & 17/3 \\ 0 & 2/3 & -4/3 \\ 0 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 6 & 13 \\ 2 & 4 & 9 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -39/10 & 21/5 & 43/10 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -79/10 & 41/5 & 83/10 \\ 2 & -2 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

Theorem 8. *For any finite matrix A, then secondary generalized inverse* $A^{\dagger\theta} = A^\theta V A^\theta$ *where* $V \in (A^\theta A A^\theta)^{(1)}$*, provided A satisfies* θ*-cancellation property.*

Proof: Now consider,

$$\begin{aligned} A^{\dagger\theta} &= A^{\dagger\theta}AA^{\dagger\theta} = A^{\dagger\theta}AA^{\dagger\theta}AA^{\dagger\theta}AA^{\dagger\theta} \\ &= A^{\dagger\theta}(AA^{\dagger\theta})^\theta A(A^{\dagger\theta}A)^\theta A^{\dagger\theta} \\ &= A^{\dagger\theta}(A^{\dagger\theta})^\theta A^\theta AA^\theta (A^{\dagger\theta})^\theta A^{\dagger\theta} \end{aligned}$$

Since $(A^\theta AA^\theta)^{(1)}$ is g-inverse of $A^\theta AA^\theta$, we have

$$\begin{aligned} A^{\dagger\theta} &= A^{\dagger\theta}(A^{\dagger\theta})^\theta A^\theta AA^\theta VA^\theta AA^\theta (A^{\dagger\theta})^\theta A^{\dagger\theta} \\ &= A^{\dagger\theta}(AA^{\dagger\theta})^\theta AA^\theta VA^\theta A(A^{\dagger\theta}A)^\theta A^{\dagger\theta} \\ &= A^{\dagger\theta}AA^{\dagger\theta}AA^\theta VA^\theta AA^{\dagger\theta}AA^{\dagger\theta} \\ &= A^{\dagger\theta}AA^\theta VA^\theta AA^{\dagger\theta} \\ &= (A^{\dagger\theta}A)A^\theta VA^\theta (AA^{\dagger\theta}) \\ &= A^\theta VA^\theta \end{aligned}$$

■

IV. Different characterizations of Secondary generalized inverse

Theorem 9. *Consider* $A \in \mathbb{C}^{m\times n}$*. Then, the following statements are equivalent.*

1) $A^{\dagger\theta}$ *exists.*
2) $rank(A^\theta AA^\theta) = rank(A)$
3) $A\mathscr{C}(A^\theta) \oplus \mathscr{N}(A) = \mathbb{C}^m$.

Proof: (1) $\Longrightarrow$ (2) If $A^{\dagger\theta}$ exists, then $rank(AA^\theta) = rank(A^\theta A) = rank(A)$.
Since $\mathscr{C}(A) \cap \mathscr{N}(A) = \{0\}$, we have

$$\begin{aligned} rank(A^\theta AA^\theta) &= rank(AA^\theta) - dim(\mathscr{C}(AA^\theta) \cap \mathscr{N}(A^\theta)) \\ &= rank(A) - dim(\mathscr{C}(A) \cap \mathscr{N}(A^\theta) \\ &= rank(A) \end{aligned}$$

(2) $\Longrightarrow$ (3) From the conditions,

$$\begin{aligned} rank(A) &= rank(A^\theta AA^\theta) \\ &= rank(AA^\theta) - dim(\mathscr{C}(AA^\theta) \cap \mathscr{N}(A^\theta)) \end{aligned}$$

and $\quad rank(A) \geq rank(AA^\theta)$

we get that $\mathscr{C}(AA^\theta) \cap \mathscr{N}(A^\theta) = \{0\}$ and $rank(AA^\theta) = rank(A)$, which implies that $A\mathscr{C}(A^\theta) \oplus \mathscr{N}(A) = \mathbb{C}^m$.
(3) $\Longrightarrow$ (1) From $A\mathscr{C}(A^\theta) \oplus \mathscr{N}(A) = \mathbb{C}^m$, it follows that $\mathscr{C}(AA^\theta) \cap \mathscr{N}(A^\theta) = \{0\}$ and $rank(AA^\theta) = rank(A)$. Thus $\mathscr{C}(A) \cap (N)(A^\theta) = \{0\}$, ie., $rank(A^\theta A) = rank(A)$. Hence $A^{\dagger\theta}$ exists. ■

We use the following important result from [11] to obtain the condition for the existence of s-g inverse in terms of the index of AA^θ and $A^\theta A$.

Theorem 10. *[11] Let* $A \in \mathbb{C}_r^{m\times n}$*, and let two subspaces* $\mathscr{T} \subseteq \mathbb{C}^n$ *and* $\mathscr{S} \in \mathbb{C}^m$ *be such that* $dim(\mathscr{T}) \leq r$ *and* $dim(\mathscr{S}) = m - dim(\mathscr{T})$*. Suppose* $H \in \mathbb{C}^{n\times m}$ *such that* $\mathscr{C}(H) = \mathscr{T}$ *and* $\mathscr{N}(H) = \mathscr{S}$*. If* $A^{(2)}_{\mathscr{T},\mathscr{S}}$ *exists, then* $Ind(AH) = 1$*. Further, we have* $A^{(2)}_{\mathscr{T},\mathscr{S}} = (HA)^\sharp H = H(AH)^\sharp$.

Theorem 11. *Consider an* $m \times n$ *matrix A. Then the following conditions are equivalent:*

1) $A^{\dagger\theta}$ *exists*
2) $Ind(A^\theta A) = 1$ *and* $\mathscr{N}(A^\theta A) \subseteq \mathscr{N}(A)$
3) $Ind(AA^\theta) = 1$ *and* $\mathscr{C}(A) \subseteq \mathscr{C}(AA^\theta)$

Proof: (1) $\Longleftrightarrow$ (2) The only if part is obvious by Theorem 10 and Lemma 1. Conversely, since $rank(A) = rank(A^\theta A)$, from $\mathscr{N}(A^\theta A) \subseteq \mathscr{N}(A)$ and $Ind(A^\theta A) = 1$, it follows that $\mathscr{C}(A^\theta) \cap \mathscr{N}(A) = \mathscr{C}(A^\theta A) \cap \mathscr{N}(A^\theta A) = \{0\}$, which implies $rank(A^\theta A) = rank(A)$. Hence $A^{\dagger\theta}$ exists directly by Lemma 1.
The proof of (1) $\Longleftrightarrow$ (3) is similar to (1) $\Longleftrightarrow$ (2).

■

Theorem 12. *Consider an* $m \times n$ *matrix with* $rank(A^\theta AA^\theta) = rank(A)$ *and let* $G \in \mathbb{C}^{n\times m}$*. Then, the following statements are equivalent:*

1) $G = A^{\dagger\theta}$
2) *There exist* $P \in \mathbb{C}^{m\times m}$ *and* $Q \in \mathbb{C}^{n\times n}$ *such that* $AGA = A, G = A^\theta P, G = QA^\theta$
   Moreover,

$$P = (A^\theta)^{(1)}A^{\dagger\theta} + (I_m - (A^\theta)^{(1)}A^\theta)B,$$

$$Q = A^{\dagger\theta}(A^\theta)^{(1)} + C(I_n - A^\theta(A^\theta)^{(1)})$$

   where $B \in \mathbb{C}^{m\times m}$ *and* $C \in \mathbb{C}^{n\times n}$ *are arbitrary and* $(A^\theta)^{(1)} \in (A^\theta)\{1\}$.

Corollary 1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $rank(A^\theta AA^\theta) = rank(A)$*, and let* $G \in \mathbb{C}^{n\times m}$*. Then, the following statements are equivalent:*

1) $G = A^{\dagger\theta}$*;*
2) *There exists* $X \in \mathbb{C}^{m\times n}$ *such that* $AGA = A, G = A^\theta XA^\theta$

In this case,

$$X = (A^\theta)^{(1)}A^{\dagger\theta}(A^\theta)^{(1)} + (I_m - (A^\theta)^{(1)}(A^\theta))P + Q(I_n - (A^\theta)(A^\theta)^{(1)})$$

where $P, Q \in \mathbb{C}^{m\times n}$ *are arbitrary and* $(A^\theta)^{(1)} \in (A^\theta)\{1\}$

Theorem 13. *Consider a matrix* $A \in \mathbb{C}^{m\times n}$*. Then, the following statements are equivalent:*

1) $A^{\dagger\theta}$ *exists;*
2) *There exists* $P \in \mathbb{C}^{m\times m}$ *such that* $A = PAA^\theta A$*;*
3) *There exists* $Q \in \mathbb{C}^{n\times n}$ *such that* $A = AA^\theta AQ$

In this case, $A^{\dagger\theta} = (PA)^\theta = (AQ)^\theta$.

Proof: There exists a $P \in \mathbb{C}^{m\times m}$ such that $A = PAA^\theta A$ is equivalent to $\mathscr{N}(AA^\theta A) \subseteq \mathscr{N}(A)$. This assertion is equivalent to $rank(A) = rank(AA^\theta A)$. Now, the equivalence of (1) and (2) is obvious by the item (2) in Theorem 9. The proof of the equivalence of (1) and (3) can be obtained in a similar way.
Moreover, if $A^{\dagger\theta}$ exists, we first claim that $(PA)^\theta \in A\{1, 3^\theta, 4^\theta\}$. In fact, using $A = PAA^\theta A$, we infer that

$$\begin{aligned} (A(PA)^\theta)^\theta &= PAA^\theta = PA(PAA^\theta A)^\theta \\ &= PAA^\theta AA^\theta P^\theta = A(PA)^\theta \end{aligned}$$

$$\begin{aligned}A(PA)^\theta A &= (A(PA)^\theta)^\theta A = PAA^\theta A = A((PA)^\theta A)^\theta = (A^\theta P^\theta A)^\theta\\ &= ((PAA^\theta A)^\theta P^\theta A)^\theta = (A^\theta AA^\theta (P^\theta)^2 A)^\theta\\ &= (A^\theta PAA^\theta A)A^\theta (P^\theta)^2 A)^\theta\\ &= (A^\theta PPAA^\theta AA^\theta AA^\theta (P^\theta)^2 A)^\theta (A^\theta (P)^2 (AA^\theta)^3 (P^\theta)^2 A)^\theta\\ &= A^\theta (P)^2 (AA^\theta)^3 (P^\theta)^2 A = (PA)^\theta A\end{aligned}$$

which implies $(PA)^\theta \in A\{1,3^\theta,4^\theta\}$.
Finally, according to Remark 1, we obtain

$$\begin{aligned}A^{\dagger\theta} &= (PA)^\theta A(PA)^\theta\\ &= ((PA)^\theta A)^\theta (PA)^\theta = A^\theta PAA^\theta P^\theta\\ &= (A(PA)^\theta A)^\theta P^\theta = (PA)^\theta\end{aligned}$$

To prove $(AY)^\theta \in A\{1,3^\theta,4^\theta\}$ and $A^{\dagger\theta} = (AQ)^\theta$ the same method given above can be used. ■

The result given in the following lemma helps to prove theorem 14.

Lemma 4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then, $I_m - AB$ is nonsingular if and only if $I_n - BA$ is non singular, and in which case, $(I_m - AB)^{-1} = I_m + A(I_n - BA)^{-1}B$*

Theorem 14. *Let $A \in \mathbb{C}^{m\times n}$ and $A^{(1)}$ is an arbitrary generalized inverse of A. Then, the following statements are equivalent:*

1) $A^{\dagger\theta}$ exists:
2) $A^\theta A + I_n - A^{(1)}A$ is nonsingular.
3) $AA^\theta + I_m - AA^{(1)}$ is nonsingular.

In this case,

$$\begin{aligned}A^{\dagger\theta} &= (A(A^\theta A + I_n - A^{(1)}A)^{-1})^\theta\\ &= ((AA^\theta + I_m - AA^{(1)})^{(-1)}A)^\theta\end{aligned}$$

Proof: Denote
$B = A^\theta A + I_n - A^{(1)}A$ and $C = AA^\theta + I_m - AA^{(1)}$.
(1) $\Longrightarrow$ (2). If $A^{\dagger\theta}$ exists, using items (1) and (2) in Theorem 13 , we have $A = PAA^\theta A$ for some $G \in \mathbb{C}^{m\times m}$.It can be easily verified that

$$(A^{(1)}PA + I_n - A^{(1)}A)(A^{(1)}AA^\theta A) + I_n - A^{(1)}A) = I_n$$

which shows the non singularity of $D = A^{(1)}AA^\theta A + I_n - A^{(1)}A$ and D can be rewritten as $D = I_n - A^{(1)}A(I_n - A^\theta A)$. Thus by lemma 4, B is non singular.
(2) $\Longrightarrow$ (1). Since B is nonsingular, from $AB = AA^\theta A$, we have $A = AA^\theta AB^{-1}$. Therefore, $A^{\dagger\theta}$ exists by items (1) and (3) of theorem 13.
(3) $\Longleftrightarrow$ (2). Since B and C can be rewritten as $B = I_n - (A^{(1)} - A^\theta)A$ and $C = I_m - A(A^{(1)} - A^\theta)$, from Lemma 4, we have the equivalence of (3) and (2) immediately.
In this case, from items (1) and (2), we infer that

$$B^\theta A^{\dagger\theta} = (A^\theta A + I_n - A^{(1)A})^\theta A^\theta$$

$$A^\theta AA^{\dagger\theta} + A^{\dagger\theta} - A^\theta (A^\theta)^{(1)} A^{\dagger\theta} = A^\theta$$

which, together with the item (2) gives $A^{\dagger\theta} - (AB^{-1})^\theta$. Analogously, we can derive that $A^{\dagger\theta} - (C^{-1}A)^\theta$. This completes the proof. ■

Theorem 15. *Let $A \in \mathbb{C}^{m\times n}$. Then, the following statements are equivalent:*

1) $A^{\dagger\theta}$ exists;
2) $rank(AA^\theta) = rank(A^\theta)$ and there exists $X \in \mathbb{C}^{m\times m}$ and a projector $Y \in \mathbb{C}^{m\times m}$ such that

$$XAA^\theta - YX = I_m,$$

$AA^\theta X = XAA^\theta$ and $AA^\theta Y = 0$. In this case, $A^{\dagger\theta} = A^\theta X$.

Proof: If $A^{\dagger\theta}$ exists, then $rank(AA^\theta) = rank(A)$ by lemma 1. Let $Q = AA^\theta + I_m - AA^{\dagger\theta}$. Therefore, $Q((A^\theta)^{\dagger\theta}A^{\dagger\theta} + I_m - AA^{\dagger\theta}) = I_m$, which implies Q is nonsingular. Also, $AA^{\dagger\theta}Q = QAA^{\dagger\theta} = AA^\theta$.
Denote $Y = I_m - AA^{\dagger\theta}$. Clearly $Y^2 = Y$ and $AA^\theta Y = YAA^\theta = 0$. Let $X = AA^{\dagger\theta}Q^{-1} - Y$. Hence

$$XAA^\theta = (AA^{\dagger\theta}Q^{-1} - Y)AA^{\dagger\theta}Q = AA^{\dagger\theta}Q^{-1}QAA^{\dagger\theta} = AA^{\dagger\theta},$$

$$-YX = -Y(AA^{\dagger\theta}Q^{-1} - Y) = -YAA^{\dagger\theta}Q^{-1} + Y = Y$$

Evidently, $XAA^\theta = AA^\theta X$ and $XAA^\theta - YX = I_m$.
(2) $\Longrightarrow$ (1). Premultiplying $XAA^\theta - YX = I_m$, by AA^θ, we have that

$$AA^\theta XAA^\theta - AA^\theta YX = AA^\theta \tag{7}$$

Equation (7) together with $AA^\theta X = XAA^\theta$ and $AA^\theta Y = 0$, gives $AA^\theta AA^\theta X = AA^\theta$ if and only if $\mathscr{C}(AA^\theta X - I_m) \subseteq \mathscr{N}(AA^\theta)$. Since $\mathscr{N}(AA^\theta) = \mathscr{N}(A^\theta)$, from $rank(AA^\theta) = rank(A^\theta)$, we get $A^\theta = A^\theta AA^\theta X$, i.e.,

$$A = X^\theta AA^\theta A.$$

Consequently, $A^{\dagger\theta}$ exists according to items (1) and (2) in Theorem 13. Finally, applying Theorem 13, we get $A^{\dagger\theta} = A^\theta X$ directly. ■

Remark 2. *Let $A \in \mathbb{C}^{m\times n}$ and there exists $X \in \mathbb{C}^{n\times n}$ and a projector $Y \in \mathbb{C}^{n\times n}$ such that $XAA^\theta - YX = I_n$, $A^\theta AX = XAA^\theta$ and $A^\theta AY = 0$. Then, $A^{\dagger\theta} = (AX)^\theta$.*

V. Conclusion

In this article, we have presented different characterizations of secondary generalized inverse and the necessary conditions for its existence.
Further, possible areas of research in this field include
1) Obtaining iterative methods for representing the secondary generalized inverse.
2) Extending the existence of secondary generalized inverse to commutative ring, Hilbert space etc.
These explorations will open newer frontiers of secondary generalized inverse.

References

[1] A. Ben-Israel and T. N. E. Greville, "Generalized Inverses:Theory and Applications," Springer, Berlin, 2003.
[2] J. Gao, K. Zuo, Q. Wang and J. Wu, "Further characterizations and representations of the Minkowski inverse in Minkowski space," *AIMS Mathematics*, vol. 8, no. 10, pp. 23403-23426, 2023.
[3] K. Kamaraj and K. C. Sivakumar, "Moore Penrose inverse in an indefinite inner product space," *J. Appl. Math.Comput.*, vol. 19, pp. 297-310, 2005.
[4] A. Lee, "Secondary symmetric, skew symmetric and orthogonal matrices," *Period. Math. Hung*, vol. 7, no. 1, pp. 63-70, 1976.
[5] V. Savitha, D. P. Shenoy, K. Umashankar and R. B. Bapat, "Secondary transpose of a matrix and generalized inverses," *Journal of Algebra and its applications*, vol. 23, no. 3, 2450052, 2024.
[6] D. P. Shenoy, "Drazin theta and theta Drazin matrices," *Numerical Algebra, Control and Optimization*, vol. 14, no. 2, pp. 273-283, 2024.
[7] D. P. Shenoy, "Drazin-Theta inverse for rectangular matrices,"*IAENG International Journal of Computer Science*, vol. 50, no. 4, pp. 1515-1521, 2023.

[8] D. P. Shenoy, "Outer Theta and Theta Outer Inverses," *IAENG International Journal of Applied Mathematics*, vol. 52, no. 4, pp. 1020-1024, 2022.
[9] R. Vijayakumar, "s-g inverse of s-normal matrices," *International Journal of Mathematics Trends and Technology*, vol. 4, no.39, pp. 240-244, 2016.
[10] D. Shenoy, "Secondary range symmetric matrices," [version 2; peer review: 2 approved, 1 approved with reservations]. *F1000Research, 2024, 13:112*, (https://doi.org/10.12688/f1000research.144171.2)
[11] Y. Wei, "A characteriation and representation of the generalized inverse $A^{2}_{T,S}$ and its applications," *Linear Algebra Appl*, vol. 280,pp. 87-96, 1998.
[12] G. Wang, Y. Wei and S. Qiao, "Generalized Inverses: Theory and Computations 2^{nd}," *B*eijing:Science Press, 2018.
[13] H. Zekraoui, Z. Al-Zhour and C. Ozel, "Some new algebraic and topological properties of the Minkowski inverse in the Minkowski space," *Sci. World J,* 765732, 2013.

Divya Shenoy P is an Assistant Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India - 576104. The author is a member of IAENG since 2023. She obtained her Ph.D from Manipal Academy of Higher Education, Manipal in the year 2017. Her research areas are generalized inverse of matrices, Drazin inverse, matrix partial orders and s-g inverse.

Pruthvi K K is a post graduate student at the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India - 576104.