Albertson*(Alb)* Energy of Product of Graphs

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***Abstract*—Albertson energy $Alb\epsilon(G)$ is defined as the sum of the absolute values of the Albertson eigenvalues of G. This paper computes the Albertson energy of various graph products, relating it to graph properties like order, degree, energy, or Albertson energy of the base graph. Our primary focus is on deriving formulas for the Albertson energy of the Cartesian, strong, and tensor products when one graph is regular. We also compute the Albertson energy of join, corona and hierarchical products of regular graphs. Furthermore, we calculate the Albertson energy of p-shadow and p-duplicate graphs.**

***Index Terms*—Irregularity measure, Albertson energy, Graph products, Shadow graph, Duplicate graph, Kronecker product.**

I. Introduction

LET $G = (V, E)$ be a simple, undirected graph. Vertices u and v in G are said to be adjacent, if $uv \in E$ and denoted as $u \sim v$. The degree of a vertex v, denoted as $deg(v)$ in G is the number of edges incident with v. A graph G is said to be regular if all its vertices have same degree. A point v is said to be the central vertex if eccentricity of v is equal to radius of G and center of G is the set of all central vertices.

Spectral graph theory revolves around the exploration of the eigenvalues associated with matrix derived from graphs. Let A be the adjacency matrix of a graph G with n vertices, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. The key outcome of this exploration is the energy of graph [1] which is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

A graph which is not regular is an irregular graph. It is natural to inquire about the extent of irregularity exhibited by an irregular graph. Hence numerous measures for quantifying irregularity have been proposed by various authors. One notable index introduced by M. O. Albertson in [2] is the Albertson index, which is defined as

$$Alb(G) = \sum_{uv \in E(G)} |deg(u) - deg(v)|.$$

It is also known as third Zagreb index. The corresponding Albertson matrix [3] $Alb(G) = [x_{ij}]$ is an $n \times n$ matrix, where

$$x_{ij} = \begin{cases} |deg(v_i) - deg(v_j)|, & \text{if } v_i \sim v_j \text{ in } G \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of $Alb(G)$ are referred to as Albertson eigenvalues of G. Consequently, Albertson energy of a graph G is defined as the sum of the absolute value of Albertson eigenvalues of G. Suppose $\xi_1 < \xi_2 < \ldots < \xi_r$, $r \leq n$ are distinct Albertson eigenvalues of G with multiplicities $m_1, m_2, \ldots m_r$, then we shall write

$$spec(Alb(G)) = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

For more information on these topics, additional studies can be found in [4]–[7]. Recent work in [8] has investigated Albertson energy of splitting graphs and shadow graphs for regular graphs. Building on these findings, we aim to extend the understanding of Albertson energy of p-shadow and p-duplicate graphs for any graph. Additionally, we examine the behavior of Albertson energy for different graph products.

Section 2 establishes fundamental definitions and results essential to prove subsequent results. In Section 3, we develop a general formula for computing the Albertson energy of Cartesian, strong, and tensor products when one of the graphs involved is regular. Additionally, we calculate the Albertson energy of the corona product and join of two regular graphs, as well as the hierarchical product of $G \sqcap P_m$ and $G \sqcap K_{1,m}$, where G is any graph, P_m is a path and $K_{1,m}$ is a star graph. Finally, Section 4 focuses on calculating Albertson energy of the p-shadow and p-duplicate graphs.

Let I_n, J_n and 0_n denote the identity matrix, the zero matrix, and the matrix with all entries equal to 1 of order n, respectively.

II. Preliminaries

Let G and H be any two graphs of order n and m respectively. The **corona product** of G and H denoted by $G \odot H$ is a graph obtained by taking one copy of G and n copies of H and joining the i^{th} vertex of G to each vertex in the i^{th} copy of H, where $i = 1, 2, \ldots, n$.

Definition 1: The **join** $G \nabla H$ of two graphs G and H is a graph obtained from joining each vertex of G to all vertices of H.

The vertex set of Cartesian product, strong product, tensor product and hierarchical product of G and H is $V(G) \times V(H)$.

Definition 2: The **Cartesian product** $G \square H$ of graphs G and H is a graph in which any two vertices (u, u') and (v, v') are adjacent if and only if either

(i) $u = v$ in G and $u' \sim v'$ in H or
(ii) $u' = v'$ in H and $u \sim v$ in G.

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Definition 3: The **strong product** $G \boxtimes H$ of graphs G and H is the graph in which any two vertices (u, u') and (v, v') are adjacent if and only if either

(i) $u = v$ in G and $u' \sim v'$ in H or
(ii) $u' = v'$ in H and $u \sim v$ in G or
(iii) $u \sim v$ in G and $u' \sim v'$ in H.

Definition 4: The **tensor product** $G \times H$ of graphs G and H is a graph in which any two vertices (u, u') and (v, v') are adjacent if and only if $u \sim v$ in G and $u' \sim v'$ in H.

Definition 5: [9] The **hierarchical product** $G \sqcap H$ of graphs G and H having a distinguished or root vertex labeled 0, in which any two vertices (u, u') and (v, v') are adjacent if and only if either

(i) $u = v$ and $u' \sim v'$ in H or
(ii) $u' = v' = 0$ and $u \sim v$ in G.

We notice that the Cartesian product, strong product and tensor product exhibit commutativity, unlike the corona product and hierarchical product.

Remark 1: By the definition of Cartesian, strong, tensor and hierarchical product, it follows that, for any $(u, v) \in V(G_1) \times V(G_2)$,

(i) $deg(u, v) = deg(u) + deg(v)$ in $G_1 \Box G_2$.
(ii) $deg(u, v) = deg(u) + deg(v) + deg(u)deg(v)$ in $G_1 \boxtimes G_2$.
(iii) $deg(u, v) = deg(u)deg(v)$ in $G_1 \times G_2$.
(iv) $deg(u, v) = \begin{cases} deg(u) + deg(v), & \text{if } v \text{ is the root vertex} \\ deg(v), & \text{otherwise.} \end{cases}$

Definition 6: The ***p*-shadow graph** $D_p(G)$ of a connected graph G is a graph constructed by creating p identical copies of G and then joining each vertex u in G_i to every neighbour of the corresponding vertex v in G_j for all $1 \leq j \leq p$.

Remark 2: Let $v_1, v_2, \ldots, v_n$ be the vertices of G and G_i be the i^{th} copy of G in $D_p(G)$, whose vertices are $v_1^{(i)}, v_2^{(i)}, \ldots, v_n^{(i)}$ such that each $v_j^{(i)}$ corresponds to vertex v_j in G for $1 \leq i \leq p$ and $1 \leq j \leq n$. We note that for each $1 \leq i \leq p$ and $1 \leq j \leq n$, $deg(v_j^{(i)})$ in $D_p(G) = p\ deg(v_j)$ in G.

Definition 7: [10] Let V' be a set such that $V \cap V' = \emptyset$, $|V| = |V'|$ and $f : V \to V'$ be bijective map (we write $f(a)$ as a'). A **duplicate graph** of G is $D(G) = (V_1, E_1)$, where the vertex set $V_1 = V \cup V'$ and the edge set E_1 of $D(G)$ is defined as the edge ab is in E if and only if both ab' and $a'b$ are in E_1. In general, $D^p(G) = D^{p-1}(D(G))$.
For (n, m) graph G, p-duplicate graph contains $2^p n$ vertices and $2^p m$ edges.

Theorem 1: [10] Let $D(G)$ be the duplicate graph of G. Then

(i) No two vertices of V and V' are adjacent.
(ii) For $a \in V$, $deg(a)$ in $G = deg(a)$ in $D(G) = deg(a')$ in $D(G)$.

Definition 8: [11] Let A and B be matrices of order $m \times n$ and $p \times q$ respectively. The **Kronecker product** of A and B, denoted by $A \otimes B$, is the $mp \times nq$ block matrix $[a_{ij}B]$.

Lemma 1: [11] Let A and B be symmetric matrices of order m and n respectively. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of A and B respectively, then the eigenvalues of $A \otimes B$ are given by $\alpha_i \mu_j$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Lemma 2: [11] Let A and B be symmetric matrices of order m and n respectively. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of A and B respectively, then the eigenvalues of $A \otimes I_n + I_m \otimes B$ are given by $\alpha_i + \mu_j$ $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Lemma 3: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of a matrix A, then the eigenvalues of $I_n + A$ will be $1 + \alpha_1, 1 + \alpha_2, \ldots, 1 + \alpha_n$.

Remark 3: If G is a regular graph, then $Alb(G)$ is equal to zero matrix.

III. ALBERTSON ENERGY OF PRODUCTS OF GRAPHS

Theorem 2: If G_1 is a k-regular graph of order n and G_2 is a r-regular graph of order m, then Albertson energy for corona product of graph G_1 and G_2 is $Alb\epsilon(G_1 \odot G_2) = 2n(k + m - r - 1)\sqrt{m}$.

Proof: Let the vertex set of G_1 and G_2 be $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_m\}$ respectively. Let $G_2^{(i)}$ denote the i^{th} copy of G_2 attached to i^{th} vertex of G_1 with vertex set $\{u_1^{(i)}, u_2^{(i)}, \ldots, u_m^{(i)}\}$.

Let the vertices in $Alb(G_1 \odot G_2)$ be listed as $v_1, v_2, \ldots, v_n, u_1^{(1)}, u_1^{(2)}, \ldots, u_1^{(n)}, u_2^{(1)}, u_2^{(2)}, \ldots, u_2^{(n)}, \ldots, u_m^{(1)}, u_m^{(2)}, \ldots, u_m^{(n)}$.
Albertson matrix of $G_1 \odot G_2$ can be written as follows:

$$Alb(G_1 \odot G_2) = \begin{bmatrix} Alb(G_1) & tI_n & \cdots & tI_n \\ tI_n & Alb(G_1) & \cdots & Alb(G_1) \\ \vdots & \vdots & \ddots & \vdots \\ tI_n & Alb(G_1) & \cdots & Alb(G_1) \end{bmatrix}_{n(m+1)},$$

where I_n is the identity matrix and $t = k + m - r - 1$.

$$Alb(G_1 \odot G_2) = \begin{bmatrix} Alb(G_1) & 0_n & \cdots & 0_n \\ 0_n & Alb(G_1) & \cdots & Alb(G_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & Alb(G_1) & \cdots & Alb(G_1) \end{bmatrix} + \begin{bmatrix} 0_n & tI_n & \cdots & tI_n \\ tI_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ tI_n & 0_n & \cdots & 0_n \end{bmatrix}.$$

As G_1 is regular, by Remark 3 we obtain
$Alb(G_1 \odot G_2) = A(K_{1,m}) \otimes tI_n$, where $K_{1,m}$ is a star graph.
We know that the spectra of $K_{1,m}$ is $\begin{pmatrix} -\sqrt{m} & 0 & \sqrt{m} \\ 1 & m-1 & 1 \end{pmatrix}$.
Using Lemma 1, we get
$Alb\epsilon(G_1 \odot G_2) = tn(|\sqrt{m}| + |-\sqrt{m}|) = 2tn\sqrt{m} = 2n(k + m - r - 1)\sqrt{m}$. ■

Remark 4: We observe that, for the graphs G_1 and G_2 as in Theorem 2, above result can also be written as
$Alb\epsilon(G_1 \odot G_2) = tnE(K_{1,m})$.

Theorem 3: If G_1 is a k-regular graph of order n and G_2 is a r-regular graph of order m, then Albertson energy for join of graph G_1 and G_2 is $Alb\epsilon(G_1 \nabla G_2) = 2|k + m - r - n|\sqrt{mn}$.

Proof: Albertson matrix of $G_1 \nabla G_2$ can be written as follows:

$$Alb(G_1 \nabla G_2) = \begin{bmatrix} 0_n & cJ_{n\times m} \\ cJ_{m\times n} & 0_m \end{bmatrix}_{n+m},$$

where $c = |k+m-r-n|$. We observe that the rank of the above block matrix is 2. Hence it has two non-zero eigenvalues, say ξ_1 and ξ_2 such that

$$\xi_1 + \xi_2 = trace(Alb(G_1 \nabla G_2)) = 0. \quad (1)$$

We have

$$(Alb(G_1 \nabla G_2))^2 = \begin{bmatrix} mc^2 J_n & 0_{n\times m} \\ 0_{m\times n} & nc^2 J_m \end{bmatrix}_{n+m}.$$

Then

$$trace((Alb(G_1 \nabla G_2))^2) = \xi_1^2 + \xi_2^2 = 2c^2 mn. \quad (2)$$

Solving Equations 1 and 2, we obtain $\xi_1 = c\sqrt{mn}$ and $\xi_2 = -c\sqrt{mn}$. Therefore,

$$Alb_\epsilon(G_1 \nabla G_2) = 2c\sqrt{mn} = 2|k+m-r-n|\sqrt{mn}.$$

■

Theorem 4: Let G_1 and G_2 be graphs of order n and m respectively. Then $0 \le Alb\epsilon(G_1 \Box G_2) \le mAlb\epsilon(G_1) + nAlb\epsilon(G_2)$. Equality holds if either G_1 or G_2 are regular.

Proof: Let the vertex set of G_1 be $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_m\}$ respectively. Let the vertices in $Alb(G_1 \Box G_2)$ be listed as $(v_1,u_1),(v_1,u_2),\ldots,(v_1,u_m),(v_2,u_1),(v_2,u_2),\ldots,$ $(v_2,u_m),\ldots,(v_n,u_1),(v_n,u_2),\ldots,(v_n,u_m)$.

Consider

$$Alb(G_1 \Box G_2) = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \ldots & X_{nn} \end{bmatrix}_{nm}, \quad (3)$$

where each $X_{ij} = [x_{kl}^{(ij)}]$ is an $m \times m$ block matrix, $1 \le i,j \le n$ and $1 \le k,l \le m$.

We claim that $Alb(G_1 \Box G_2) = (Alb(G_1) \otimes I_m) + (I_n \otimes Alb(G_2))$, which is equivalent to proving that

$$X_{ij} = \begin{cases} Alb(G_2), & \text{if } i = j \\ |deg(v_i) - deg(v_j)| I_m, & \text{if } v_i \sim v_j \text{ in } G_1 \\ 0_m, & \text{otherwise.} \end{cases}$$

Case 1: Consider the diagonal matrix X_{ii}, $1 \le i \le n$.
The corresponding row and column indices of X_{ii} is $\{(v_i,u_1),(v_i,u_2),\ldots,(v_i,u_m)\}$. For any $1 \le k,l \le m$,

$$x_{kl}^{(ii)} = \begin{cases} |deg(v_i,u_k) - deg(v_i,u_l)|, & \text{if } (v_i,u_k) \sim (v_i,u_l) \\ 0, & \text{otherwise.} \end{cases}$$

By the definition of Cartesian product, we have $(v_i,u_k) \sim (v_i,u_l)$ in $G_1 \Box G_2 \implies u_k \sim u_l$ in G_2. Also from Remark 1, $x_{kl}^{(ii)} = |deg(v_i,u_k) - deg(v_i,u_l)| = |deg(u_k) - deg(u_l)|$.

$\implies X_{ii} = [x_{kl}^{(ii)}] = Alb(G_2),\ 1 \le i \le n,\ 1 \le k,l \le m$.

Case 2: Consider the non-diagonal matrix X_{ij}, $1 \le i,j \le n$. The corresponding row and column indices of X_{ij} are $\{(v_i,u_1),(v_i,u_2),\ldots,(v_i,u_m)\}$ and $\{(v_j,u_1),(v_j,u_2),\ldots,(v_j,u_m)\}$ respectively.

For any $1 \le k,l \le m$,

$$x_{kl}^{(ij)} = \begin{cases} |deg(v_i,u_k) - deg(v_j,u_l)|, & \text{if } (v_i,u_k) \sim (v_j,u_l) \\ 0, & \text{otherwise.} \end{cases}$$

First, we shall consider the case where $v_i \sim v_j$ in G_1.
By the definition of Cartesian product, we observe that $(v_i,u_k) \sim (v_j,u_l)$ in $G_1 \Box G_2 \implies u_k = u_l$ in G_2. By Remark 1, $x_{kk}^{(ij)} = |deg(v_i,u_k) - deg(v_j,u_k)| = |deg(v_i) - deg(v_j)|$.

$\implies X_{ij} = [x_{kl}^{(ij)}] = |deg(v_i) - deg(v_j)| I_m,\ 1 \le i \le n,\ 1 \le k,l \le m$, whenever $v_i \sim v_j$.

Now we will consider the case where $v_i \nsim v_j$ in G_1.
We observe that if $v_i \nsim v_j$ in G_1, then $(v_i,u_k) \nsim (v_j,u_l)$ in $G_1 \Box G_2$.

$\implies X_{ij} = [x_{kl}^{(ij)}] = 0_m,\ 1 \le i \le n,\ 1 \le k,l \le m$, whenever $v_i \nsim v_j$.

Therefore, it follows that $Alb(G_1 \Box G_2) = (Alb(G_1) \otimes I_m) + (I_n \otimes Alb(G_2))$. Let $\xi_i,\ 1 \le i \le n$ and $\xi'_j,\ 1 \le j \le m$ be the Albertson eigenvalues of G_1 and G_2 respectively. Then by Lemma 2, we obtain

$$\begin{aligned} Alb\epsilon(G_1 \Box G_2) &= \sum_{i=1}^{n} \sum_{j=1}^{m} |\xi_i + \xi'_j| \\ &\le \sum_{i=1}^{n} m|\xi_i| + \sum_{j=1}^{m} n|\xi'_j| \\ &= mAlb\epsilon(G_1) + nAlb\epsilon(G_2). \end{aligned}$$

$$\begin{aligned} Alb\epsilon(G_1 \Box G_2) &= \sum_{i=1}^{n} \sum_{j=1}^{m} |\xi_i + \xi'_j| \\ &\ge \Big| \sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_i + \xi'_j) \Big| \\ &= \Big| m\sum_{i=1}^{n} \xi_i + n\sum_{j=1}^{m} \xi'_j \Big| \\ &= 0. \end{aligned}$$

Therefore, $0 \le Alb\epsilon(G_1 \Box G_2) \le mAlb\epsilon(G_1) + nAlb\epsilon(G_2)$. Equality follows from Remark 3 and Theorem 4. ■

Theorem 5: Let G be a graph of order n. Then $2Alb\epsilon(G) \le Alb\epsilon(G \Box G) \le 2nAlb\epsilon(G)$. Equality holds if G is regular.

Proof: The proof is similar to that in Theorem 4. We obtain $Alb(G \Box G) = (Alb(G) \otimes I_n) + (I_n \otimes Alb(G))$. Let $\xi_i,\ 1 \le i \le n$ be Albertson eigenvalues of G. Then by Lemma 2, we obtain

$$\begin{aligned} Alb\epsilon(G \Box G) &= \sum_{i,j=1}^{n} |\xi_i + \xi_j| \\ &\le 2\sum_{i=1}^{n} |\xi_i| + \sum_{\substack{i,j=1 \\ i \ne j}}^{n} |\xi_i| + |\xi_j| \\ &= 2Alb\epsilon(G) + 2(n-1)Alb\epsilon(G) \\ &= 2nAlb\epsilon(G). \end{aligned}$$

$$\begin{aligned} Alb\epsilon(G \Box G) &\ge 2Alb\epsilon(G) + \Big| \sum_{i,j=1}^{n} (\xi_i + \xi_j) \Big| \\ &= 2Alb\epsilon(G) + 2(n-1)trace(Alb(G)) \\ &= 2Alb\epsilon(G). \end{aligned}$$

Therefore, $2Alb\epsilon(G) \leq Alb\epsilon(G\square G) \leq 2nAlb\epsilon(G)$. ■

Theorem 6: 4 Let G_1 be r-regular graph of order n and G_2 be any graph of order m. Then Albertson energy for Cartesian product of G_1 and G_2 is $Alb\epsilon(G_1\square G_2) = nAlb\epsilon(G_2)$.

Proof: Let the vertex set of G_1 and G_2 be $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_m\}$ respectively. Let the vertices in $Alb(G_1\square G_2)$ be listed as $(v_1, u_1), (v_1, u_2), \ldots, (v_1, u_m), (v_2, u_1), (v_2, u_2), \ldots,$ $(v_2, u_m), \ldots, (v_n, u_1), (v_n, u_2), \ldots, (v_n, u_m)$.
Let $Alb(G_1\square G_2)$ be the matrix as in Equation 3.
Case 1: Consider the diagonal matrix X_{ii}, $1 \leq i \leq n$.
As in Theorem 4, we obtain $X_{ii} = [x_{kl}^{(ii)}] = Alb(G_2),\ 1 \leq i \leq n,\ 1 \leq k, l \leq m$.
Case 2: Consider the non-diagonal matrix $X_{ij}, 1 \leq i, j \leq n$. The corresponding row and column indices of X_{ij} are $\{(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_m)\}$ and $\{(v_j, u_1), (v_j, u_2), \ldots, (v_j, u_m)\}$ respectively.
For any $1 \leq k, l \leq m$,

$$x_{kl}^{(ij)} = \begin{cases} |deg(v_i, u_k) - deg(v_j, u_l)|, & \text{if } \ (v_i, u_k) \sim (v_j, u_l) \\ 0, & \text{otherwise.} \end{cases}$$

By the definition of Cartesian product, we observe that $(v_i, u_k) \sim (v_j, u_l)$ in $G_1\square G_2 \implies u_k = u_l$ in G_2. This implies $x_{kl}^{(ij)} = |deg(v_i, u_k) - deg(v_j, u_l)| = 0$.
Hence, $X_{ij} = [x_{kl}^{(ij)}] = 0_{m\times m},\ 1 \leq i, j \leq n,\ 1 \leq k, l \leq m$.
Therefore, we obtain

$$Alb(G_1\square G_2) = \begin{bmatrix} Alb(G_2) & 0_m & \cdots & 0_m \\ 0_m & Alb(G_2) & \cdots & 0_m \\ \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \ldots & Alb(G_2) \end{bmatrix}_{nm},$$

is a block diagonal matrix.

Hence, Albertson eigenvalues of the above matrix is equal to the eigenvalues of each diagonal matrices. Suppose $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_m$ are Albertson eigenvalues of G_2. Then Albertson energy of $G_1\square G_2$ is

$$Alb\epsilon(G_1\square G_2) = n\sum_{i=1}^{m} |\xi_i| = nAlb\epsilon(G_2).$$

■

Theorem 7: Let G_1 be r-regular graph of order n and G_2 be any graph of order m. Then Albertson energy for strong product of G_1 and G_2 is
$Alb\epsilon(G_1 \boxtimes G_2) = (r+1)Alb\epsilon(G_2)\sum_{j=1}^{n} |(1+\lambda_j)|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A(G_1)$.

Proof: Let the vertex set of G_1 and G_2 be $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_m\}$ respectively. Let the vertices in $Alb(G_1 \boxtimes G_2)$ be listed as $(v_1, u_1), (v_1, u_2), \ldots, (v_1, u_m), (v_2, u_1), (v_2, u_2), \ldots,$ $(v_2, u_m), \ldots, (v_n, u_1), (v_n, u_2), \ldots, (v_n, u_m)$.
Let $Alb(G_1 \boxtimes G_2)$ be the matrix as in Equation 3. We claim that the $Alb(G_1 \boxtimes G_2) = (I_n + A(G_1)) \otimes (r+1)Alb(G_2)$, which is equivalent to proving that for any $1 \leq i, j \leq n$,

$$X_{ij} = \begin{cases} (r+1)Alb(G_2), & \text{if } i = j \\ (r+1)Alb(G_2), & \text{if } v_i \sim v_j \text{ in } G_1 \\ 0_m, & \text{otherwise.} \end{cases}$$

Case 1: Consider the diagonal matrix X_{ii}, $1 \leq i \leq n$.
The corresponding row and column indices of X_{ii} is $\{(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_m)\}$. For any $1 \leq k, l \leq m$,

$$x_{kl}^{(ii)} = \begin{cases} |deg(v_i, u_k) - deg(v_i, u_l)|, & \text{if } \ (v_i, u_k) \sim (v_i, u_l) \\ 0, & \text{otherwise.} \end{cases}$$

By the definition of strong product, we observe that $(v_i, u_k) \sim (v_i, u_l)$ in $G_1 \boxtimes G_2$ implies that $u_k \sim u_l$ in G_2. Also by Remark 1,
$x_{kl}^{(ii)} = |deg(v_i, u_k) - deg(v_i, u_l)| = (r+1)|deg(u_k) - deg(u_l)|$.
$\implies X_{ii} = [x_{kl}^{(ii)}] = (r+1)Alb(G_2),\ 1 \leq i \leq n,\ 1 \leq k, l \leq m$.
Case 2: Consider the non-diagonal matrix X_{ij}, $1 \leq i, j \leq n$. The corresponding row and column indices of X_{ij} are $\{(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_m)\}$ and $\{(v_j, u_1), (v_j, u_2), \ldots, (v_j, u_m)\}$ respectively.
For any $1 \leq k, l \leq m$,

$$x_{kl}^{(ij)} = \begin{cases} |deg(v_i, u_k) - deg(v_j, u_l)|, & \text{if } \ (v_i, u_k) \sim (v_j, u_l) \\ 0, & \text{otherwise.} \end{cases}$$

First, we shall consider the case where $v_i \sim v_j$ in G_1.
By the definition of strong product, we observe that $(v_i, u_k) \sim (v_j, u_l)$ in $G_1 \boxtimes G_2$ and $v_i \sim v_j$ in $G_1 \implies u_k \sim u_l$ or $u_k = u_l$ in G_2. Hence, $x_{kl}^{(ij)} = |deg(v_i, u_k) - deg(v_j, u_l)| = (r+1)|deg(u_k) - deg(u_l)|$ whenever $u_k \sim u_l$ and $x_{kk}^{(ij)} = 0$.
$\implies X_{ij} = [x_{kl}^{(ij)}] = (r+1)Alb(G_2),\ 1 \leq i, j \leq n,$ $1 \leq k, l \leq m,$ whenever $v_i \sim v_j$.
Now we will consider the case where $v_i \nsim v_j$ in G_1.
We observe that if $v_i \nsim v_j$ in G_1, then $(v_i, u_k) \nsim (v_j, u_l)$ in $G_1 \boxtimes G_2$.
Hence, $X_{ij} = [x_{kl}^{(ij)}] = 0_m,\ 1 \leq i, j \leq n,\ 1 \leq k, l \leq m,$ whenever $v_i \nsim v_j$. Therefore, it follows that

$$Alb(G_1 \boxtimes G_2) = (I_n + A(G_1)) \otimes (r+1)Alb(G_2).$$

Let $\xi_i,\ 1 \leq i \leq m$ and $\lambda_j,\ 1 \leq j \leq n$ be the eigenvalues of $Alb(G_2)$ and $A(G_1)$ respectively. By applying Lemma 1 and 3, we obtain

$$\begin{aligned} Alb\epsilon(G_1 \boxtimes G_2) &= \sum_{j=1}^{n}\sum_{i=1}^{m} |(r+1)\xi_i(1+\lambda_j)| \\ &= (r+1)\sum_{j=1}^{n}\sum_{i=1}^{m} |\xi_i||(1+\lambda_j)| \\ &= (r+1)Alb\epsilon(G_2)\sum_{j=1}^{n} |(1+\lambda_j)|. \end{aligned}$$

■

Theorem 8: Let G_1 be r-regular graph of order n and G_2 be any graph of order m. Then Albertson energy for tensor product of G_1 and G_2 is $Alb\epsilon(G_1 \times G_2) = rE(G_1)Alb\epsilon(G_2)$.

Proof: Let the vertex set of G_1 and G_2 be $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_m\}$ respectively. Let the vertices in $Alb(G_1 \times G_2)$ be listed as $(v_1, u_1), (v_1, u_2), \ldots, (v_1, u_m), (v_2, u_1), (v_2, u_2), \ldots,$ $(v_2, u_m), \ldots, (v_n, u_1), (v_n, u_2), \ldots, (v_n, u_m)$.
Let $Alb(G_1 \times G_2)$ be the matrix as in Equation 3. We claim

that the $Alb(G_1 \times G_2) = rA(G_1) \otimes Alb(G_2)$, which is equivalent to proving that for any $1 \le i, j \le n$,

$$X_{ij} = \begin{cases} 0_m, & \text{if } i = j \\ rAlb(G_2), & \text{if } v_i \sim v_j \text{ in } G_1 \\ 0_m, & \text{otherwise.} \end{cases}$$

Case 1: Consider the diagonal matrix X_{ii}, $1 \le i \le n$.
The corresponding row and column indices of X_{ii} is $\{(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_m)\}$.
We note that in $G_1 \times G_2$ $(v_i, u_k) \nsim (v_i, u_l)$, for $1 \le k, l \le m$, since we are dealing with simple graphs.
$\implies X_{ii} = [x_{kl}^{(ii)}] = 0_{m\times m},\ 1 \le i \le n,\ 1 \le k, l \le m.$
Case 2: Consider the non-diagonal matrix $X_{ij}, 1 \le i, j \le n$.
The corresponding row and column indices of X_{ij} are $\{(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_m)\}$ and $\{(v_j, u_1), (v_j, u_2), \ldots, (v_j, u_m)\}$ respectively.
For any $1 \le k, l \le m$,

$$x_{kl}^{(ij)} = \begin{cases} |deg(v_i, u_k) - deg(v_j, u_l)|, & \text{if } (v_i, u_k) \sim (v_j, u_l) \\ 0, & \text{otherwise.} \end{cases}$$

Consider the case where $v_i \sim v_j$ in G_1.
By the definition of tensor product, we observe that $(v_i, u_k) \sim (v_i, u_l)$ in $G_1 \times G_2$ and $v_i \sim v_j$ in $G_1 \implies u_k \sim u_l$ in G_2. By Remark 1, $x_{kl}^{(ij)} = |deg(v_i, u_k) - deg(v_j, u_l)| = r|deg(u_k) - deg(u_l)|$.
$\implies X_{ij} = [x_{kl}^{(ij)}] = rAlb(G_2),\ 1 \le i, j \le n,\ 1 \le k, l \le m,$ whenever $v_i \sim v_j$.
Now, consider the case where $v_i \nsim v_j$ in G_1.
This implies that $(v_i, u_k) \nsim (v_j, u_l)$ in $G_1 \times G_2$.
$\implies X_{ij} = [x_{kl}^{(ij)}] = 0_{m\times m},\ 1 \le i, j \le n,\ 1 \le k, l \le m,$ whenever $v_i \nsim v_j$.
Let $\xi_i,\ 1 \le i \le m$ and $\lambda_j,\ 1 \le j \le n$ be the eigenvalues of $Alb(G_2)$ and $A(G_1)$ respectively. From Lemma 1, Albertson energy of $G_1 \times G_2$ is

$$\begin{aligned} Alb\epsilon(G_1 \times G_2) &= \sum_{j=1}^{n}\sum_{i=1}^{m} |r\xi_i\lambda_j| \\ &= r\sum_{j=1}^{n}|\lambda_j|\sum_{i=1}^{m}|\xi_i| \\ &= rE(G_1)Alb\epsilon(G_2). \end{aligned}$$

■

Theorem 9: Albertson energy of hierarchical product of a r-regular graph G of order n and a path P_m with a pendent vertex as the root is

$$Alb\epsilon(G \sqcap P_m) \begin{cases} 2n\sqrt{r^2 - 2r + 2}, & \text{if } m = 3 \\ 2nr, & \text{if } m > 3. \end{cases}$$

Proof: Let the vertex set of G and P_m be $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ respectively, where u_1 and u_m are pendent vertices and $u_i \sim u_{i+1}$ for $1 \le i \le m - 1$. Without loss of generality, assume that u_1 is the root vertex. Let the vertices in $Alb(G \sqcap P_m)$ be listed as $(v_1, u_1), (v_2, u_1), \ldots, (v_n, u_1), (v_1, u_2), (v_2, u_2), \ldots, (v_n, u_2), \ldots, (v_1, u_m), (v_2, u_m), \ldots, (v_n, u_m)$.
Albertson matrix of $G \sqcap P_m$ can be written as follows:

$$Alb(G \sqcap P_m) = \begin{bmatrix} \mathcal{N} & (r-1)I_n & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} \\ (r-1)I_n & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} \\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & I_n \\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & I_n & \mathcal{N} \end{bmatrix}_{nm},$$

where $\mathcal{N} = Alb(G)$. As G is a regular graph, from Remark 3, we obtain $Alb(G \sqcap P_m) = C \otimes I_n$, where

$$C = \begin{bmatrix} 0 & (r-1) & 0 & \cdots & 0 & 0 \\ (r-1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_m.$$

For $m > 3$, the characteristic polynomial of C is $det(\alpha I - C) = \alpha^{m-4}(\alpha^2 - 1)(\alpha^2 - (r-1)^2)$ and $Spec(C) = \begin{pmatrix} -r+1 & -1 & 0 & 1 & r-1 \\ 1 & 1 & m-4 & 1 & 1 \end{pmatrix}$.
By using Lemma 1, we get

$$\begin{aligned} Alb\epsilon(G \sqcap P_m) &= |n(-r+1)| + |-n| + |n| + |n(r-1)| \\ &= 2nr. \end{aligned}$$

For $m = 3$, we have $C = \begin{bmatrix} 0 & r-1 & 0 \\ r-1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Then the characteristic polynomial of C is $det(\alpha I - C) = \alpha(\alpha^2 - (r^2 - 2r + 2))$ and $Spec(C) = \begin{pmatrix} -\sqrt{r^2-2r+2} & 0 & \sqrt{r^2-2r+2} \\ 1 & 1 & 1 \end{pmatrix}$.
From Lemma 1, we get

$$\begin{aligned} Alb\epsilon(G \sqcap P_m) &= |n\sqrt{r^2 - 2r + 2}| + |-n\sqrt{r^2 - 2r + 2}| \\ &= 2n\sqrt{r^2 - 2r + 2}. \end{aligned}$$

■

Remark 5: In the above theorem, for $m = 2$, we get $Alb(G \sqcap P_2) = r\begin{bmatrix} 0_n & I_n \\ I_n & 0_n \end{bmatrix}$. Therefore $Alb_\epsilon(G \sqcap P_2) = 2nr$.

Theorem 10: Albertson energy of the hierarchical product of a r-regular graph G of order n and a path P_m with an odd value of m and the central vertex as the root is

$$Alb\epsilon(G \sqcap P_m) = \begin{cases} 2n\sqrt{2}(r+1), & \text{if } m = 3 \\ 2n(1 + \sqrt{2r^2 + 1}), & \text{if } m = 5. \\ 2n(2 + \sqrt{2}r), & \text{if } m > 5 \end{cases}$$

Proof: Let the vertex set of G and P_m be $V(G) = \{v_1,\ v_2, \ldots, v_n\}$ and $V(P_m) = \{u_1,\ u_2, \ldots, u_m\}$ respectively, where u_1 and u_m are pendent vertices and $u_i \sim u_{i+1}$ for $1 \le i \le m - 1$. Let $u_{\frac{m+1}{2}}$ be the root vertex of P_m. Let the vertices in $Alb(G \sqcap P_m)$ be listed as follows: $(v_1, u_1),\ (v_2, u_1), \ldots, (v_n, u_1),\ (v_1, u_2),\ (v_2, u_2), \ldots, (v_n, u_2), \ldots, (v_1, u_m), (v_2, u_m), \ldots, (v_n, u_m)$.
Albertson matrix of $G \sqcap P_m$ can be written as follows:

$$Alb(G\sqcap P_m)=\begin{bmatrix}\mathcal{N} & I_n & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N}\\ I_n & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N}\\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & rI_n & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N}\\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & rI_n & \mathcal{N} & rI_n & \cdots & \mathcal{N} & \mathcal{N}\\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & rI_n & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & I_n\\ \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} & \mathcal{N} & \cdots & I_n & \mathcal{N}\end{bmatrix}_{nm},$$

where $\mathcal{N} = Alb(G)$. As G is a regular graph, from Remark 3, we obtain $Alb(G \sqcap P_m) = E \otimes I_n$, where

$$E=\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & r & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & r & 0 & r & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & r & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0\end{bmatrix}_m.$$

For $m > 5$, the characteristic polynomial of E is $det(\alpha I - E) = \alpha^{m-6}(\alpha^2 - 1)^2(\alpha^2 - 2r^2)$ and $Spec(E) = \begin{pmatrix} -\sqrt{2}r & -1 & 0 & 1 & \sqrt{2}r \\ 1 & 2 & m-6 & 2 & 1 \end{pmatrix}$.
By using Lemma 1, we get

$$\begin{aligned} Alb\epsilon(G \sqcap P_m) &= |n\sqrt{2}r| + 2|-n| + 2|n| + |-n\sqrt{2}r| \\ &= 2n(2 + \sqrt{2}r). \end{aligned}$$

For $m = 3$, we have $E = \begin{bmatrix} 0 & r+1 & 0 \\ r+1 & 0 & r+1 \\ 0 & r+1 & 0 \end{bmatrix}$.
The characteristic polynomial of E is $det(\alpha I - E) = \alpha(\alpha^2 - 2(r+1)^2)$ and $Spec(E) = \begin{pmatrix} -\sqrt{2}(r+1) & 0 & \sqrt{2}(r+1) \\ 1 & 1 & 1 \end{pmatrix}$.
By using Lemma 1, we get

$$\begin{aligned} Alb\epsilon(G \sqcap P_m) &= |n\sqrt{2}(r+1)| + |-n\sqrt{2}(r+1)| \\ &= 2n\sqrt{2}(r+1). \end{aligned}$$

For $m = 5$, we have $E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & r & 0 & 0 \\ 0 & r & 0 & r & 0 \\ 0 & 0 & r & 0 & 1 \\ 0 & 0 & r & 1 & 0 \end{bmatrix}$.
The characteristic polynomial of E is $det(\alpha I - E) = \alpha(\alpha^2 - 1)(\alpha^2 - 2r^2 - 1)$ and $Spec(E) = \begin{pmatrix} -\sqrt{2r^2+1} & -1 & 0 & 1 & \sqrt{2r^2+1} \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.
By using Lemma 1, we get

$$Alb\epsilon(G \sqcap P_m) = 2n(1 + \sqrt{2r^2 + 1}).$$

■

Theorem 11: Albertson energy of hierarchical product of a r-regular graph G of order n and a star graph $K_{1,m-1}$ with central vertex as the root is $Alb\epsilon(G \sqcap K_{1,m-1}) = n(r+m-2)E(K_{1,m-1})$.

Proof: Let the vertex set of G and $K_{1,m-1}$ be $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_{1,m-1}) = \{u_1, u_2, \ldots, u_m\}$ respectively, with u_1 as the central vertex and hence the root vertex. Let the vertices in $Alb(G \sqcap K_{1,m-1})$ be listed as $(v_1, u_1), (v_2, u_1), \ldots, (v_n, u_1), (v_1, u_2), (v_2, u_2), \ldots, (v_n, u_2), \ldots, (v_1, u_m), (v_2, u_m), \ldots, (v_n, u_m)$.
Albertson matrix of $G \sqcap P_m$ can be written as follows:

$$Alb(G \sqcap K_{1,m-1}) = \begin{bmatrix} \mathcal{N} & \mathcal{M} & \mathcal{M} & \cdots & \mathcal{M} \\ \mathcal{M} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} \\ \mathcal{M} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} \end{bmatrix}_{nm},$$

where $\mathcal{N} = Alb(G)$ and $\mathcal{M} = (r + m - 2)I_n$. As G is a regular graph, from Remark 3, we obtain
$Alb(G \sqcap K_{1,m-1}) = A(K_{1,m-1}) \otimes (r + m - 2)I_n$.
The graph $K_{1,m-1}$ has eigenvalues $\sqrt{m-1}$, $-\sqrt{m-1}$ and 0 with multiplicity 1, 1 and $m-2$ respectively. From Lemma 1, Albertson energy of $G \sqcap K_{1,m-1}$ is

$$\begin{aligned} Alb\epsilon(G \sqcap K_{1,m-1}) &= (r + m - 2)\sum_{i=1}^{m} 2n\sqrt{m-1} \\ &= n(r + m - 2)E(K_{1,m-1}). \end{aligned}$$

■

Theorem 12: Albertson energy of the hierarchical product of a r-regular graph G of order n and a star graph $K_{1,m-1}$ with any pendent vertex as the root is $Alb\epsilon(G \sqcap K_{1,m-1}) = 2n\sqrt{(m-r-2)^2 + (m-2)^3}$.

Proof: Let the vertex set of G and $K_{1,m-1}$ be $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_{1,m-1}) = \{u_1, u_2, \ldots, u_m\}$ respectively, with u_1 as the central vertex. Without loss of generality, assume that u_2 is the root vertex. Let the vertices in $Alb(G \sqcap K_{1,m-1})$ be listed as $(v_1, u_1), (v_2, u_1), \ldots, (v_n, u_1), (v_1, u_2), (v_2, u_2), \ldots, (v_n, u_2), \ldots, (v_1, u_m), (v_2, u_m), \ldots, (v_n, u_m)$.
Albertson matrix of $G \sqcap K_{1,m-1}$ can be written as follows:
$Alb\epsilon(G \sqcap K_{1,m-1}) =$

$$\begin{bmatrix} Alb(G) & |m-r-2|I_n & (m-2)I_n & \cdots & (m-2)I_n \\ |m-r-2|I_n & Alb(G) & Alb(G) & \cdots & Alb(G) \\ (m-2)I_n & Alb(G) & Alb(G) & \cdots & Alb(G) \\ (m-2)I_n & Alb(G) & Alb(G) & \cdots & Alb(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-2)I_n & Alb(G) & Alb(G) & \cdots & Alb(G) \end{bmatrix}_{nm}.$$

As G is a regular graph, from Remark 3, we obtain
$Alb\epsilon(G \sqcap K_{1,m-1}) = F \otimes I_n$, where

$$F = \begin{bmatrix} 0 & |m-r-2| & (m-2) & \cdots & (m-2) \\ |m-r-2| & 0 & 0 & \cdots & 0 \\ (m-2) & 0 & 0 & \cdots & 0 \\ (m-2) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-2) & 0 & 0 & \cdots & 0 \end{bmatrix}_m.$$

We note that rank of F is 2. Thus F has two nonzero eigenvalues say α_1 and α_2. Hence

$$\alpha_1 + \alpha_2 = trace(F) = 0. \tag{4}$$

$F^2 =$

$$\left[\begin{array}{c|c|c} [(m-r-2)^2+(m-2)^3]J_1 & 0_1 & 0_{1\times m-2} \\ \hline 0_1 & (m-r-2)^2 J_1 & cJ_{1\times m-2} \\ \hline 0_{m-2\times 1} & cJ_{m-2\times 1} & (m-2)^2 J_{m-2} \end{array}\right],$$

where $c = (m-2)|m-r-2|$.

$$trace(F^2) = \alpha_1^2 + \alpha_2^2 = 2|m-r-2|^2 + 2(m-2)^3 \quad (5)$$

Solving Equations 4 and 5, we obtain
$\alpha_1 = \sqrt{|m-r-2|^2 + (m-2)^3}$
and $\alpha_2 = -\sqrt{|m-r-2|^2 + (m-2)^3}$.
By applying Lemma 1, we obtain
$Alb\epsilon(G \sqcap K_{1,m-1}) = 2n\sqrt{(m-r-2)^2 + (m-2)^3}$. ■

IV. Albertson energy of p-Shadow and p-duplicate graphs

In this section, we consider graph G of order n.

Theorem 13: Albertson energy of the p-shadow graph $D_p(G)$ of G is $Alb\epsilon(D_p(G)) = p^2 Alb\epsilon(G)$.

Proof: Albertson matrix of $D_p(G)$ can be written as

$$Alb(D_p(G)) = \begin{bmatrix} pAlb(G) & pAlb(G) & \cdots & pAlb(G) \\ pAlb(G) & pAlb(G) & \cdots & pAlb(G) \\ \vdots & \vdots & \ddots & \vdots \\ pAlb(G) & pAlb(G) & \cdots & pAlb(G) \end{bmatrix}_{pn}$$
$$= J_p \otimes pAlb(G).$$

Rank of the matrix J_p is 1. Hence J_p has exactly one non-zero eigenvalue, which is equal to $trace(J_p) = p$.
Suppose $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$ are Albertson eigenvalues of G. Then by using Lemma 1, we obtain

$$Alb\epsilon(D_p(G)) = \sum_{i=1}^{n} |p^2\xi_i| = p^2 \sum_{i=1}^{n} |\xi_i| = p^2 Alb\epsilon(G).$$

■

Remark 6: By the Theorem 1, we observe that for a graph G with n vertices, Albertson matrix of $D(G)$ is

$$Alb(D(G)) = \begin{bmatrix} 0_n & Alb(G) \\ Alb(G) & 0_n \end{bmatrix}_{2n}$$

Theorem 14: Albertson energy of the p-duplicate graph $D^p(G)$ of G is $Alb\epsilon(G) = 2^p Alb\epsilon(G)$.

Proof: Albertson matrix of $D^p(G)$ can be written as

$$Alb(D^p(G)) = \begin{bmatrix} 0_n & 0_n & \cdots & 0_n & Alb(G) \\ 0_n & 0_n & \cdots & Alb(G) & 0_n \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0_n & Alb(G) & \cdots & 0_n & 0_n \\ Alb(G) & 0_n & \cdots & 0_n & 0_n \end{bmatrix}_{2^p n}$$
$$= K \otimes Alb(G),$$

where $K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2^p}$.

The eigenvalues of K are 1 and -1, each with multiplicity 2^{p-1}. Suppose $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$ are the Albertson eigenvalues of G. Then by using Lemma 1, we obtain

$$Alb\epsilon(D^p(G)) = \sum_{i=1}^{n} (2^{p-1}|\xi_i| + 2^{p-1}|-\xi_i|) = 2^p Alb\epsilon(G).$$

■

References

[1] I. Gutman, "The energy of a graph," *Ber. Math.–Statist. Sekt. Forschungszentrum Graz*, vol. 103, p. 1–22, 1978.

[2] M. O. Albertson, "The irregularity of a graph," *Ars Comb.*, vol. 46, pp. 219–225, 1997.

[3] A. Jahanbani, "Albertson energy and albertson estrada index of graphs," *J. Linear. Topological. Algebra.*, vol. 8, no. 01, pp. 11–24, 2019.

[4] A. Ghalavand, I. Gutman, M. Tavakoli *et al.*, "Irregularity measure of graphs," *Journal of Mathematics*, vol. 2023, 2023.

[5] H. Abdo and D. Dimitrov, "The irregularity of graphs under graph operations," *Discussiones Mathematicae Graph Theory*, vol. 34, no. 2, pp. 263–278, 2014.

[6] I. Gutman, "Irregularity of molecular graphs," *Kragujevac J. Sci.*, no. 38, pp. 71–81, 2016.

[7] S. Nayak, S. D'souza, and P. G. Bhat, "Color laplacian energy of generalised complements of a graph," *Engineering Letters*, vol. 29, no. 4, pp. 1502–1510, 2021.

[8] M. M. Munir and U. T. Wusqa, "Albertson (alb) spectral radii and albertson (alb) energies of graph operation," *Front. Chem.*, vol. 11, 2023.

[9] L. Barriere, F. Comellas, C. Dalfó, and M. A. Fiol, "The hierarchical product of graphs," *Discrete Appl. Math.*, vol. 157, no. 1, pp. 36–48, 2009.

[10] E. Sampathkumar, "On duplicate graphs," *J. Indian Math. Soc*, vol. 37, pp. 285–293, 1973.

[11] R. B. Bapat, *Graphs and matrices*. London: Springer, 2010, vol. 27.