Event-triggered Stabilization for Neural Networks Subject to Replay Attacks

Yuxiang Ji, Yu Zhang, Ling Chen, and Jianping Zhou

***Abstract*—This paper addresses the issue of event-triggered stabilization for neural networks vulnerable to replay attacks. According to the proposed event triggering mechanism and controller, the neural network is represented as a switched closed-loop system. Through the selection of a suitable Lyapunov function and applying Jensen's inequality, a criterion to ensure the mean square exponential stability of the system is established in the form of linear matrix inequalities. Building on this, a co-design method for the event trigger matrix and controller gain is given. Finally, a numerical example is used to verify the effectiveness of the method.**

***Index Terms*—Neural network, event-triggered control, replay attack, exponential stability.**

I. Introduction

SINCE Walter Pitts and Warren McCulloch first proposed the concept of neural networks (NNs) in the early 20th century [1], NNs have been utilized in various fields such as image classification [2, 3], associative memory [4], image encryption [5], and pattern recognition [6, 7]. Stability is crucial for most applications, yet NNs often exhibit unstable phenomena such as chaos [8], oscillation [9], and bifurcation [10]. Therefore, maintaining the stability of NNs has been a primary research focus. To meet this goal, numerous studies have proposed a range of control strategies, including impulse control [11], non-fragile control [12], fixed-time periodic control [13], PID control [14], and sampled-data control [15, 16].

Within the automation community, event-triggered control (ETC), as a refinement of sampled-data control, has become increasingly popular because it ensures ideal control performance and reduces the overuse of communication channels in digital networks [17–19]. In this strategy, the event generator sends a signal to the controller only when system state changes exceed a preset threshold. With this approach, ETC effectively lowers the frequency of controller updates, conserving computational and network resources. Presently, a vast array of literature focuses on maintaining NN stability via ETC; see, e.g., [20–25].

However, as a networked control scheme, the event generator is susceptible to network attacks during data transmission. A common attack, the replay attack, involves an attacker recording the information sequence from the sensor and maliciously substituting channel data with previously captured legitimate data to disrupt system operation [26, 27]. Notably, the once sensational Stuxnet virus successfully exploited such an attack [28]. Consequently, defending against replay attacks has emerged as an urgent issue, requiring further research and innovative solutions.

Inspired by the preceding analysis and discussions, this paper aims to solve the event-triggered stabilization problem for NNs under replay attacks. Through the proposed event triggering mechanism (ETM) and controller, the NN is remodeled as a switched closed-loop system (CLS). Based on the constructed Lyapunov function and utilizing Jensen 's inequality, a theorem guaranteeing the mean square exponential (MSE) stability of the CLS is given in the form of linear matrix inequalities. Subsequently, the design method of the event trigger matrix and the controller gain are presented based on the proposed theorem. Finally, the effectiveness of the results is verified by a numerical example.

The remainder of this paper is organized as follows: Section II outlines the NN model, ETM, replay attack model, and the issue to be addressed. Section III details the main results, and Section IV provides an example to illustrate the effectiveness of the proposed method. Finally, Section V offers the conclusion.

Notation. Throughout this paper, $col\{\cdot\}$ represents a column vector, $\mathbb{E}\{\cdot\}$ indicates the expectation operator, and $diag\{\cdot\}$ denotes a block-diagonal matrix. $Z > 0$ indicates that Z is symmetric positive definite, $\mathcal{H}e\{Z\}$ represents the sum of Z and its transpose Z^T, and $\lambda_{min}(Z)$ ($\lambda_{max}(Z)$) denotes its minimum (maximum) eigenvalue.

II. Preliminaries

A. NN model

Consider a NN as follows

$$\dot{\zeta}(t) = A\zeta(t) + u(t) + B_h h(\zeta(t)), \tag{1}$$

in which $\zeta(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state and control input; $A \in \mathbb{R}^{n\times n}$ and $B_h \in \mathbb{R}^{n\times n}$ are known NN matrices. $h(\cdot) = col\{h_1(\cdot), h_2(\cdot), \cdots, h_n(\cdot)\} \in \mathbb{R}^n$ is a nonlinear function that satisfies the following assumption:

Assumption 1. *The nonlinear function $h_i(\cdot)$ is continuous and bounded if it fulfills $h_i(0) = 0$ and the following condition:*

$$0 \leq \frac{h_i(p_2) - h_i(p_1)}{p_2 - p_1} \leq j_i,\ i = 1, 2, \cdots, n,$$

in which p_1, $p_2 \in \mathbb{R}$, $p_1 \neq p_2$, and $\eta_i > 0$ is a constant. To simplify notation, we define $J = diag\{j_1, j_2, \cdots, j_n\}$.

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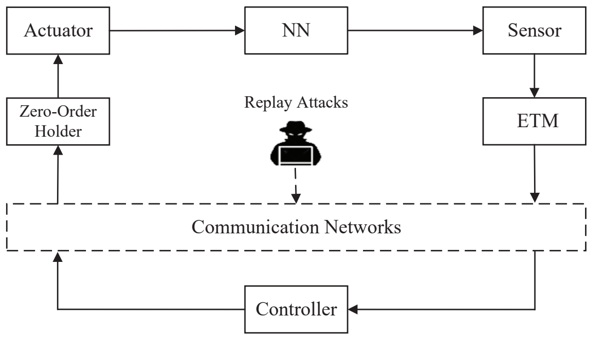

Fig. 1. The ETC framework for NNs under replay attacks.

B. ETM

To save limited network communication resources, we propose an ETM as follows:

$$t_{l+1} = min\{t \geq t_l + h \,|\, [\zeta(t) - \zeta(t_l)]^T \Phi[\zeta(t) - \zeta(t_l)] > \sigma\zeta^T(t)\Phi\zeta(t)\}, \quad (2)$$

where t_l denotes the most recent trigger moment, h represents the sample interval, σ means the trigger threshold, and Φ indicates the event trigger matrix.

C. Replay attacks

As shown in Fig. 1, we assume that the network communication channel is vulnerable to replay attacks. In this case, an attacker can intercept and record the signals in the channel. When a replay attack occurs, the attacker replaces the transmitted signal with the previously recorded signals. Based on the above discussion, the signal received by the controller can be expressed as

$$\zeta(t) = \alpha(t)\zeta(t_r) + (1 - \alpha(t))\zeta(t_l), \quad (3)$$

where $\zeta(t_r)$ represents the maliciously replaced signal. $\alpha(t)$ is a Bernoulli variable, which indicates whether a replay attack occurs.

D. Formulation of issue

According to the presented ETM, the controller under replay attacks can be written as:

$$u(t) = \alpha(t)K\zeta(t_r) + (1 - \alpha(t))K\zeta(t_l), \quad (4)$$

in which K denotes the controller gain.

Subsequently, we define $\Xi_l^1 = [t_l, t_l + h)$ and $\Xi_l^2 = [t_l + h, t_{l+1})$, under controller (4), NN (1) can be re-described as:

$$\begin{cases} \dot{\zeta}(t) = [A + (1 - \alpha(t))K]\zeta(t) + \alpha(t)K\zeta(t_r) \\ \qquad + B_h h(\zeta(t)) - (1 - \alpha(t))K \int_{t_l}^{t} \dot{\zeta}(s)ds, \; t \in \Xi_l^1, \\ \dot{\zeta}(t) = [A + (1 - \alpha(t))K]\zeta(t) + \alpha(t)K\zeta(t_r) \\ \qquad + B_h h(\zeta(t)) + (1 - \alpha(t))Ke(t), \; t \in \Xi_l^2, \end{cases} \quad (5)$$

where $e(t) = \zeta(t_l) - \zeta(t)$ fulfilling

$$e^T(t)\Phi e(t) \leq \sigma\zeta^T(t)\Phi\zeta(t). \quad (6)$$

Therefore, the issue regarding event-triggered stabilization in response to replay attacks can be summarized as follows: for a given NN (1), design controller (4) such that switched CLS (5) is mean square exponentially stable under all acceptable replay attacks.

III. Main results

In this section, we first give the MSE stability criterion for switched CLS (5), then introduce the co-design method of the event trigger matrix and controller gain.

A. Stability analysis

Theorem 1. *For given scalars $\xi \in (0, \infty)$, $h \in (0, \infty)$, $\hat{\alpha} \in [0, 1)$, $\sigma \in [0, 1)$, and matrices K, $J > 0$, if there exist matrices $\Phi > 0$, $P > 0$, $R > 0$, S_1, S_2, Q_j, F_j $(j = 1, 2, 3)$, and diagonal matrix $M > 0$ such that*

$$\tilde{\Lambda}_0^1 = \begin{bmatrix} \tilde{\Pi}_{11}^{01} & \tilde{\Pi}_{12}^{01} & \tilde{\Pi}_{13}^{01} & \tilde{\Pi}_{14}^{01} & \tilde{\Pi}_{15}^{01} \\ * & \tilde{\Pi}_{22}^{01} & \tilde{\Pi}_{23}^{01} & F_2^T B_h & \tilde{\Pi}_{25}^{01} \\ * & * & \tilde{\Pi}_{33}^{01} & 0 & 0 \\ * & * & * & -\mathcal{H}e\{M\} & B_h^T F_3 \\ * & * & * & * & \tilde{\Pi}_{55}^{01} \end{bmatrix} < 0, \quad (7)$$

$$\tilde{\Lambda}_0^2 = \begin{bmatrix} \tilde{\Pi}_{11}^{02} & \tilde{\Pi}_{12}^{02} & \tilde{\Pi}_{13}^{02} & \tilde{\Pi}_{14}^{02} & \tilde{\Pi}_{15}^{02} & \tilde{\Pi}_{16}^{02} \\ * & \tilde{\Pi}_{22}^{02} & Q_2^T & F_2^T B_h & \tilde{\Pi}_{25}^{02} & \tilde{\Pi}_{26}^{02} \\ * & * & \tilde{\Pi}_{33}^{02} & 0 & 0 & hQ_3^T \\ * & * & * & -\mathcal{H}e\{M\} & B_h^T F_3 & 0 \\ * & * & * & * & \tilde{\Pi}_{55}^{02} & \tilde{\Pi}_{56}^{02} \\ * & * & * & * & * & \tilde{\Pi}_{66}^{02} \end{bmatrix} < 0, \quad (8)$$

$$\tilde{\Lambda}_1 = \begin{bmatrix} \tilde{\Pi}_{11}^{1} & \tilde{\Pi}_{12}^{1} & \tilde{\Pi}_{13}^{1} & \tilde{\Pi}_{14}^{1} & \tilde{\Pi}_{15}^{1} \\ * & \tilde{\Pi}_{22}^{1} & \tilde{\Pi}_{23}^{1} & F_2^T B_h & \tilde{\Pi}_{25}^{1} \\ * & * & -\Phi & 0 & \tilde{\Pi}_{35}^{1} \\ * & * & * & -\mathcal{H}e\{M\} & B_h^T F_3 \\ * & * & * & * & \tilde{\Pi}_{55}^{1} \end{bmatrix} < 0 \quad (9)$$

hold, where

$$\tilde{\Pi}_{11}^{01} = 2\xi P + \mathcal{H}e\{\frac{(2\xi h - 1)}{2}S_1 + F_1^T(A + (1 - \hat{\alpha})K) - Q_1\},$$
$$\tilde{\Pi}_{12}^{01} = P + \frac{h}{2}\mathcal{H}e\{S_1\} - Q_2 - F_1 + (A + (1 - \hat{\alpha})K)^T F_2,$$
$$\tilde{\Pi}_{13}^{01} = (2\xi h - 1)(S_2 - S_1) + Q_1^T - Q_3,$$
$$\tilde{\Pi}_{14}^{01} = F_1^T B_h + J^T M^T,$$
$$\tilde{\Pi}_{15}^{01} = \hat{\alpha}F_1^T K + (A + (1 - \hat{\alpha})K)^T F_3, \tilde{\Pi}_{22}^{01} = hR - \mathcal{H}e\{F_2\},$$
$$\tilde{\Pi}_{23}^{01} = h(S_2 - S_1) + Q_2^T, \; \tilde{\Pi}_{25}^{01} = \hat{\alpha}F_2^T K - F_3,$$
$$\tilde{\Pi}_{33}^{01} = (2\xi h - 1)\mathcal{H}e\{\frac{S_1}{2} - S_2\} + \mathcal{H}e\{Q_3\},$$
$$\tilde{\Pi}_{55}^{01} = \mathcal{H}e\{\hat{\alpha}F_3^T K\},$$
$$\tilde{\Pi}_{11}^{02} = 2\xi P + \mathcal{H}e\{-\frac{S_1}{2} - Q_1 + F_1^T(A + (1 - \hat{\alpha})K)\},$$
$$\tilde{\Pi}_{12}^{02} = P - Q_2 - F_1 + (A + (1 - \hat{\alpha})K)^T F_2,$$
$$\tilde{\Pi}_{13}^{02} = S_1 - S_2 + Q_1^T - Q_3, \; \tilde{\Pi}_{14}^{02} = F_1^T B_h + J^T M^T,$$
$$\tilde{\Pi}_{15}^{02} = \hat{\alpha}F_1^T K + (A + (1 - \hat{\alpha})K)^T F_3,$$
$$\tilde{\Pi}_{16}^{02} = hQ_1^T - (1 - \hat{\alpha})hF_1^T K, \tilde{\Pi}_{22}^{02} = \mathcal{H}e\{-F_2\},$$
$$\tilde{\Pi}_{25}^{02} = \hat{\alpha}F_2^T K - F_3, \; \tilde{\Pi}_{26}^{02} = hQ_2^T - (1 - \hat{\alpha})hF_2^T K,$$
$$\tilde{\Pi}_{33}^{02} = \mathcal{H}e\{-\frac{S_1}{2} + S_2 + Q_3\}, \; \tilde{\Pi}_{55}^{02} = \mathcal{H}e\{\hat{\alpha}F_3^T K\},$$
$$\tilde{\Pi}_{56}^{02} = -(1 - \hat{\alpha})hF_3^T K, \; \tilde{\Pi}_{66}^{02} = -he^{-2\xi h}R,$$
$$\tilde{\Pi}_{11}^{1} = 2\xi P + \mathcal{H}e\{F_1^T(A + (1 - \hat{\alpha})K)\} + \sigma\Phi,$$
$$\tilde{\Pi}_{12}^{1} = P - F_1 + (A + (1 - \hat{\alpha})K)^T F_2,$$
$$\tilde{\Pi}_{13}^{1} = (1 - \hat{\alpha})F_1^T K, \; \tilde{\Pi}_{14}^{1} = F_1^T B_h + J^T M^T,$$
$$\tilde{\Pi}_{15}^{1} = \hat{\alpha}F_1^T K + (A + (1 - \hat{\alpha})K)^T F_3,$$
$$\tilde{\Pi}_{22}^{1} = \mathcal{H}e\{-F_2\}, \; \tilde{\Pi}_{23}^{1} = (1 - \hat{\alpha})F_2^T K,$$

$\tilde{\Pi}^1_{25} = \hat{\alpha}F_2^TK - F_3$, $\tilde{\Pi}^1_{35} = (1-\hat{\alpha})K^TF_3$,
$\tilde{\Pi}^1_{55} = \mathcal{H}e\{\hat{\alpha}F_3^TK\}$.

Then, switched CLS (5) is mean square exponentially stable under all acceptable replay attacks.

Proof: *We select the following Lyapunov function:*

$$\mathcal{V}(t) = \begin{cases} \mathcal{V}_0(t) = \mathcal{V}_P(t) + \mathcal{V}_R(t) + \mathcal{V}_S(t),\ t \in \Xi^1_l, \\ \mathcal{V}_1(t) = \mathcal{V}_P(t),\ t \in \Xi^2_l, \end{cases} \tag{10}$$

where

$$\begin{aligned} &\mathcal{V}_P(t) = \zeta^T(t)P\zeta(t), \\ &\mathcal{V}_R(t) = (t_l + h - t)\int_{t_l}^t e^{-2\xi(t-s)}\dot{\zeta}^T(s)R\dot{\zeta}(s)ds, \\ &\mathcal{V}_S(t) = (t_l+h-t)\begin{bmatrix}\zeta(t)\\ \zeta(t_l)\end{bmatrix}^T\begin{bmatrix}\frac{\mathcal{H}e\{S_1\}}{2} & -S_1+S_2 \\ * & \mathcal{H}e\{\frac{S_1}{2}-S_2\}\end{bmatrix}\begin{bmatrix}\zeta(t)\\ \zeta(t_l)\end{bmatrix}. \end{aligned}$$

According to the constructed function (10), we can conclude that:

$$\mathcal{V}(t) \geq \lambda_{min}(P)\|\zeta(t)\|^2,\ \mathcal{V}(0) \leq \lambda_{max}(P)\|\zeta(0)\|^2. \tag{11}$$

When $t \in \Xi^1_l$, by calculating the derivation of $\mathcal{V}_0(t)$ and taking its mathematical expectations, we can derive:

$$\mathbb{E}\{\dot{\mathcal{V}}_0(t)\} = \mathbb{E}\{\dot{\mathcal{V}}_P(t)\} + \mathbb{E}\{\dot{\mathcal{V}}_R(t)\} + \mathbb{E}\{\dot{\mathcal{V}}_S(t)\},$$

where

$$\begin{aligned} \mathbb{E}\{\dot{\mathcal{V}}_P(t)\} =& \mathbb{E}\{-2\xi\mathcal{V}_P(t)\} + 2\xi\zeta^T(t)P\zeta(t) + 2\zeta^T(t)P\dot{\zeta}(t), \\ \mathbb{E}\{\dot{\mathcal{V}}_R(t)\} =& \mathbb{E}\{-2\xi\mathcal{V}_R(t)\} - \int_{t_l}^t e^{-2\xi(t-s)}\dot{\zeta}^T(s)R\dot{\zeta}(s)ds \\ &+ (t_l+h-t)\dot{\zeta}^T(t)R\dot{\zeta}(t), \\ \mathbb{E}\{\dot{\mathcal{V}}_S(t)\} =& \mathbb{E}\{-2\xi\mathcal{V}_S(t)\} + (t_l+h-t)[\dot{\zeta}^T(t)\mathcal{H}e\{S_1\}\zeta(t) \\ &+ 2\dot{\zeta}^T(t)(-S_1+S_2)\zeta(t_l)] + [2\xi(t_l+h-t)-1] \\ &\times\left[\zeta^T(t)\frac{\mathcal{H}e\{S_1\}}{2}\zeta(t) + 2\zeta^T(t)(-S_1+S_2)\zeta(t_l)\right. \\ &\left.+ \zeta^T(t_l)\mathcal{H}e\left\{\frac{S_1}{2}-S_2\right\}\zeta(t_l)\right]. \end{aligned}$$

Therefore, we can obtain

$$\begin{aligned} \mathbb{E}\{\dot{\mathcal{V}}_0(t)\} \leq& \mathbb{E}\{-2\xi\mathcal{V}_0(t)\} + 2\xi\zeta^T(t)P\zeta(t) \\ &+ 2\zeta^T(t)P\dot{\zeta}(t) + (t_l+h-t)\dot{\zeta}^T(t)R\dot{\zeta}(t) \\ &- e^{-2\xi h}\int_{t_l}^t \dot{\zeta}^T(s)R\dot{\zeta}(s)ds + (t_l+h-t) \\ &\times[\zeta^T(t)\mathcal{H}e\{S_1\}\dot{\zeta}(t) + 2\dot{\zeta}^T(t)(-S_1+S_2)\zeta(t_l)] \\ &+ [2\xi(t_l+h-t)-1]\left[\zeta^T(t)\frac{\mathcal{H}e\{S_1\}}{2}\zeta(t)\right. \\ &+ 2\zeta^T(t)(-S_1+S_2)\zeta(t_l) \\ &\left.+ \zeta^T(t_l)\mathcal{H}e\left\{\frac{S_1}{2}-S_2\right\}\zeta(t_l)\right]. \end{aligned} \tag{12}$$

Defining

$$\varphi(t) = \frac{1}{t-t_l}\int_{t_l}^t \dot{\zeta}(s)ds.$$

Then, using Jensen's inequality [29], we can find

$$-\int_{t_l}^t \dot{\zeta}^T(s)R\dot{\zeta}(s)ds \leq -(t-t_l)\varphi^T(t)R\varphi(t).$$

On the basis of Assumption 1, for any diagonal matrix $M > 0$, we can deduce that

$$0 \leq -2h^T(\zeta(t))M[h(\zeta(t)) - J\zeta(t)]. \tag{13}$$

In addition, by applying the Newton-Leibniz theorem and CLS (5), we can derive

$$\begin{aligned} 0 =& 2\left[\zeta^T(t)Q_1^T + \dot{\zeta}^T(t)Q_2^T + \zeta^T(t_l)Q_3^T\right] \\ &\times[-\zeta(t) + \zeta(t_l) + (t-t_l)\varphi(t)], \end{aligned} \tag{14}$$

$$\begin{aligned} 0 =& 2\left[\zeta^T(t)F_1^T + \dot{\zeta}^T(t)F_2^T + \zeta^T(t_r)F_3^T\right] \\ &\times\left[-\dot{\zeta}(t) + (A + (1-\alpha(t))K)\zeta(t) + \alpha(t)K\zeta(t_r)\right. \\ &+ B_hh(\zeta(t)) - (1-\alpha(t))K(t-t_l)\varphi(t)]. \end{aligned} \tag{15}$$

Combining (12)-(15), we can obtain

$$\begin{aligned} &\mathbb{E}\{\dot{\mathcal{V}}_0(t)\} + 2\xi\mathbb{E}\{\mathcal{V}_0(t)\} \\ &\leq \frac{t_l+h-t}{h}\mu_1^T(t)\tilde{\Lambda}^1_0\mu_1(t) + \frac{t-t_l}{h}\mu_2^T(t)\tilde{\Lambda}^2_0\mu_2(t), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mu_1(t) &= [\zeta(t),\ \dot{\zeta}(t),\ \zeta(t_l),\ h(\zeta(t)),\ \zeta(t_r)], \\ \mu_2(t) &= [\zeta(t),\ \dot{\zeta}(t),\ \zeta(t_l),\ h(\zeta(t)),\ \zeta(t_r),\ \varphi(t)]. \end{aligned}$$

Based on (16) and conditions $\tilde{\Lambda}^1_0 < 0$ and $\tilde{\Lambda}^2_0 < 0$, we can establish

$$\mathbb{E}\{\dot{\mathcal{V}}_0(t)\} + 2\xi\mathbb{E}\{\mathcal{V}_0(t)\} \leq 0. \tag{17}$$

When $t \in \Xi^2_l$, using methods similar to the previous proof, we can easily get

$$\begin{aligned} \mathbb{E}\{\dot{\mathcal{V}}_2(t)\} \leq& -2\xi\mathbb{E}\{\mathcal{V}_2(t)\} + 2\xi\zeta^T(t)P\zeta(t) + 2\zeta^T(t)P\dot{\zeta}(t) \\ &+ 2[\zeta^T(t)F_1^T + \dot{\zeta}^T(t)F_2^T + \zeta^T(t_r)F_3^T] \\ &\times\left[-\dot{\zeta}(t) + (A + (1-\hat{\alpha})K)\zeta(t)\right. \\ &+ \hat{\alpha}K\zeta(t_r) + (1-\hat{\alpha})Ke(t) + B_hh(\zeta(t))] \\ &- e^T(t)\Phi e(t) + \sigma\zeta^T(t)\Phi\zeta(t) \\ &- 2h^T(\zeta(t))M[h(\zeta(t)) - J\zeta(t)] \\ \leq& \mu_3^T(t)\tilde{\Lambda}_1\mu_3(t), \end{aligned} \tag{18}$$

where

$$\mu_3(t) = [\zeta(t),\ \dot{\zeta}(t),\ e(t),\ h(\zeta(t)),\ \zeta(t_r)].$$

Combining (18) with condition $\tilde{\Lambda}_1 < 0$, we can deduce that

$$\mathbb{E}\{\dot{\mathcal{V}}_1(t)\} + 2\xi\mathbb{E}\{\mathcal{V}_1(t)\} \leq 0. \tag{19}$$

Through the constructed Lyapunov function $\mathcal{V}(t)$, it is not difficult to get

$$\begin{aligned} &\mathcal{V}_R(t_l) = \mathcal{V}_S(t_l) = 0, \\ &\lim_{t\to(t_l+h)^-}\mathcal{V}_R(t) = \lim_{t\to(t_l+h)^-}\mathcal{V}_S(t) = 0, \end{aligned}$$

which proves the continuity of $\mathcal{V}(t)$ at moments t_l and t_l+h. Then, for any $t \in \Xi^1_l$, it follows from (17) and (19) that

$$\begin{aligned} \mathbb{E}\{\mathcal{V}(t)\} &\leq e^{-2\xi(t-t_l)}\mathbb{E}\{\mathcal{V}(t_l)\} \\ &\leq e^{-2\xi(t-t_{l-1})}\mathbb{E}\{\mathcal{V}(t_{l-1})\} \\ &\ \ \cdots \\ &\leq e^{-2\xi t}\mathbb{E}\{\mathcal{V}(0)\}. \end{aligned} \tag{20}$$

Similarly, for any $t \in \Xi_l^2$*, we can obtain the same result as in (20).*

Therefore, for any $t \in \Xi_l^1 \cup \Xi_l^2$*, the following inequality always holds:*

$$\mathbb{E}\{\mathcal{V}(t)\} \leq e^{-2\xi t}\mathbb{E}\{\mathcal{V}(0)\},$$

which, in conjunction with (11), implies

$$\mathbb{E}\{\|\zeta(t)\|\} \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}e^{-\xi t}\mathbb{E}\{\|\zeta(0)\|\}. \quad (21)$$

The proof is completed. ■

B. Controller synthesis

Based on Theorem 1, we give the co-design method of the controller gain and event trigger matrix.

Theorem 2. *For given scalars* $\xi \in (0, \infty)$*,* $h \in (0, \infty)$*,* $\varrho_1 \in (0, \infty)$*,* $\varrho_2 \in (0, \infty)$*,* $\hat{\alpha} \in [0, 1)$*,* $\sigma \in [0, 1)$ *and matrix* $J > 0$*, if there exist matrices* $\Phi > 0$*,* $P > 0$*,* $R > 0$*,* F_1*,* S_1*,* S_2*,* X*,* Q_j $(j = 1, 2, 3)$ *and diagonal matrix* $M > 0$ *such that*

$$\bar{\Lambda}_0^1 = \begin{bmatrix} \bar{\Pi}_{11}^{01} & \bar{\Pi}_{12}^{01} & \bar{\Pi}_{13}^{01} & \bar{\Pi}_{14}^{01} & \bar{\Pi}_{15}^{01} \\ * & \bar{\Pi}_{22}^{01} & \bar{\Pi}_{23}^{01} & \varrho_1 F_1^T B_h & \bar{\Pi}_{25}^{01} \\ * & * & \bar{\Pi}_{33}^{01} & 0 & \bar{\Pi}_{35}^{01} \\ * & * & * & -\mathcal{H}e\{M\} & B_h^T F_3 \\ * & * & * & * & \bar{\Pi}_{55}^{01} \end{bmatrix} < 0, \quad (22)$$

$$\bar{\Lambda}_0^2 = \begin{bmatrix} \bar{\Pi}_{11}^{02} & \bar{\Pi}_{12}^{02} & \bar{\Pi}_{13}^{02} & \bar{\Pi}_{14}^{02} & \bar{\Pi}_{15}^{02} & \bar{\Pi}_{16}^{02} \\ * & \bar{\Pi}_{22}^{02} & Q_2^T & \varrho_1 F_1^T B_h & \bar{\Pi}_{25}^{02} & \bar{\Pi}_{26}^{02} \\ * & * & \bar{\Pi}_{33}^{02} & 0 & 0 & hQ_3^T \\ * & * & * & -\mathcal{H}e\{M\} & \varrho_2 B_h^T F_1 & 0 \\ * & * & * & * & \bar{\Pi}_{55}^{02} & \bar{\Pi}_{56}^{02} \\ * & * & * & * & * & \tilde{\Pi}_{66}^{02} \end{bmatrix} < 0, \quad (23)$$

$$\bar{\Lambda}_1 = \begin{bmatrix} \bar{\Pi}_{11}^{1} & \bar{\Pi}_{12}^{1} & \bar{\Pi}_{13}^{1} & \bar{\Pi}_{14}^{1} & \bar{\Pi}_{15}^{1} \\ * & \bar{\Pi}_{22}^{1} & \bar{\Pi}_{23}^{1} & \varrho_1 F_1^T B_h & \bar{\Pi}_{25}^{1} \\ * & * & -\Phi & 0 & \bar{\Pi}_{35}^{1} \\ * & * & * & -\mathcal{H}e\{M\} & \varrho_2 B_h^T F_1 \\ * & * & * & * & \bar{\Pi}_{55}^{1} \end{bmatrix} < 0 \quad (24)$$

hold, where

$$\begin{aligned}
\bar{\Pi}_{11}^{01} =& 2\xi P + \frac{(2\xi h - 1)}{2}\mathcal{H}e\{S_1\} \\
&+ \mathcal{H}e\{F_1^T A + (1 - \hat{\alpha})X - Q_1\}, \\
\bar{\Pi}_{12}^{01} =& P + \frac{h}{2}\mathcal{H}e\{S_1\} - Q_2 - F_1 \\
&+ \varrho_1 A^T F_1 + (1 - \hat{\alpha})\varrho_1 X^T, \\
\bar{\Pi}_{13}^{01} =& (2\xi h - 1)(S_2 - S_1) + Q_1^T - Q_3, \\
\bar{\Pi}_{14}^{01} =& F_1^T B_h + J^T M^T, \\
\bar{\Pi}_{15}^{01} =& \hat{\alpha}X + \varrho_2 A^T F_1 + (1 - \hat{\alpha})\varrho_2 X^T, \\
\bar{\Pi}_{22}^{01} =& hR - \mathcal{H}e\{\varrho_1 F_1\}, \\
\bar{\Pi}_{23}^{01} =& h(S_2 - S_1) + Q_2^T, \ \bar{\Pi}_{25}^{01} = \hat{\alpha}\varrho_1 X - \varrho_2 F_1, \\
\bar{\Pi}_{33}^{01} =& (2\xi h - 1)\mathcal{H}e\{\frac{S_1}{2} - S_2\} + \mathcal{H}e\{Q_3\}, \\
\bar{\Pi}_{55}^{01} =& \mathcal{H}e\{\hat{\alpha}\varrho_2 X\}, \\
\bar{\Pi}_{11}^{02} =& 2\xi P + \mathcal{H}e\{-\frac{S_1}{2} - Q_1 + F_1^T A + (1 - \hat{\alpha})X\}, \\
\bar{\Pi}_{12}^{02} =& P - Q_2 - F_1 + \varrho_1 A^T F_1 + (1 - \hat{\alpha})\varrho_1 X^T, \\
\bar{\Pi}_{13}^{02} =& S_1 - S_2 + Q_1^T - Q_3, \ \bar{\Pi}_{14}^{02} = F_1^T B_h + J^T M^T, \\
\bar{\Pi}_{15}^{02} =& \hat{\alpha}X + \varrho_2 A^T F_1 + (1 - \hat{\alpha})\varrho_2 X^T, \\
\bar{\Pi}_{16}^{02} =& hQ_1^T - (1 - \hat{\alpha})hX, \ \bar{\Pi}_{22}^{02} = \mathcal{H}e\{-\varrho_1 F_1\}, \\
\bar{\Pi}_{25}^{02} =& \hat{\alpha}\varrho_1 X - \varrho_2 F_1, \ \bar{\Pi}_{26}^{02} = hQ_2^T - (1 - \hat{\alpha})h\varrho_1 X, \\
\bar{\Pi}_{33}^{02} =& \mathcal{H}e\{-\frac{S_1}{2} + S_2 + Q_3\}, \ \bar{\Pi}_{55}^{02} = \mathcal{H}e\{\hat{\alpha}\varrho_2 X\}, \\
\bar{\Pi}_{56}^{02} =& -(1 - \hat{\alpha})h\varrho_2 X, \ \tilde{\Pi}_{66}^{02} = -he^{-2\xi h}R, \\
\bar{\Pi}_{11}^{1} =& 2\xi P + \mathcal{H}e\{F_1^T A + (1 - \hat{\alpha})X\} + \sigma\Phi, \\
\bar{\Pi}_{12}^{1} =& P - F_1 + \varrho_1 A^T F_1 + (1 - \hat{\alpha})\varrho_1 X^T, \\
\bar{\Pi}_{13}^{1} =& (1 - \hat{\alpha})X, \ \bar{\Pi}_{14}^{1} = F_1^T B_h + J^T M^T, \\
\bar{\Pi}_{15}^{1} =& \hat{\alpha}X + \varrho_2 A^T F_1 + (1 - \hat{\alpha})\varrho_2 X^T, \\
\bar{\Pi}_{22}^{1} =& \mathcal{H}e\{-\varrho_1 F_1\}, \ \bar{\Pi}_{23}^{1} = (1 - \hat{\alpha})\varrho_1 X, \\
\bar{\Pi}_{25}^{1} =& \hat{\alpha}\varrho_1 X - \varrho_2 F_1, \ \bar{\Pi}_{35}^{1} = (1 - \hat{\alpha})\varrho_2 X^T, \\
\bar{\Pi}_{55}^{1} =& \mathcal{H}e\{\hat{\alpha}\varrho_2 X\}.
\end{aligned}$$

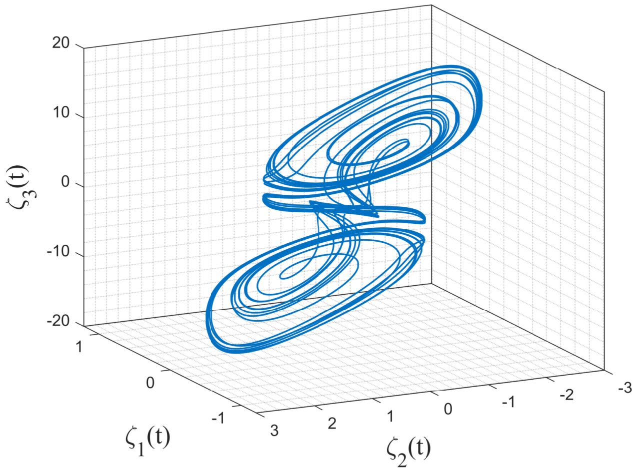

Fig. 2. Chaotic behavior.

Then, when the event trigger matrix is Φ *and the controller gain is*

$$K = (F_1^T)^{-1}X, \quad (25)$$

switched CLS (5) is mean square exponentially stable under all acceptable replay attacks.

Proof: *We can rewrite (25) as follows:*

$$X = F_1^T K. \quad (26)$$

Define

$$F_2 = \varrho_1 F_1, \ F_3 = \varrho_2 F_1. \quad (27)$$

Then, substituting (26) and (27) into (22)-(24) results in

$$\tilde{\Lambda}_0^1 < 0, \ \tilde{\Lambda}_0^2 < 0, \ \tilde{\Lambda}_1 < 0,$$

which implies (7)-(9), respectively. The proof is completed. ■

IV. Numerical Example

In this section, we use a numerical example to illustrate the effectiveness of the method presented in Theorem 2.

Consider a Hopfield NN with the following parameters [30]:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ B_h = \begin{bmatrix} 1.5 & 1.995 & 0.995 \\ -2.1 & 1.68 & 0 \\ 3.977 & -18 & 1.97 \end{bmatrix},$$

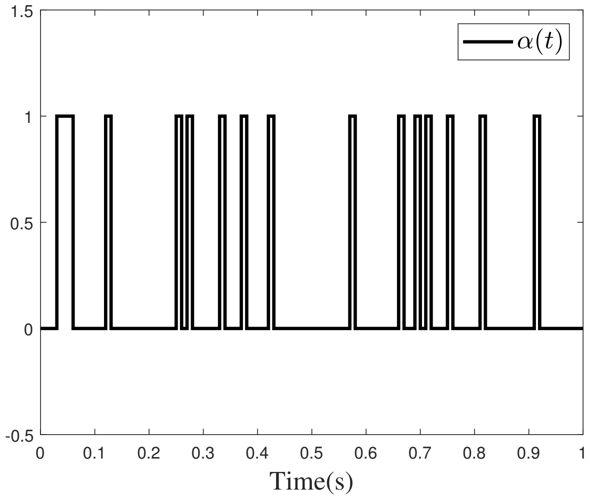

Fig. 3. Bernoulli variable $\alpha(t)$.

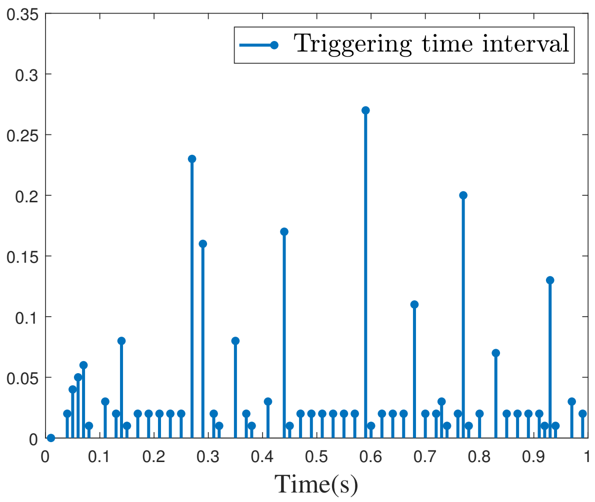

Fig. 4. Triggering instants and triggering intervals.

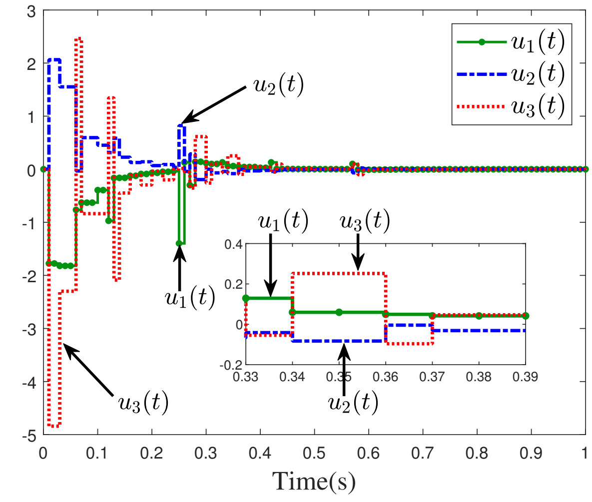

Fig. 5. The trajectories of control inputs.

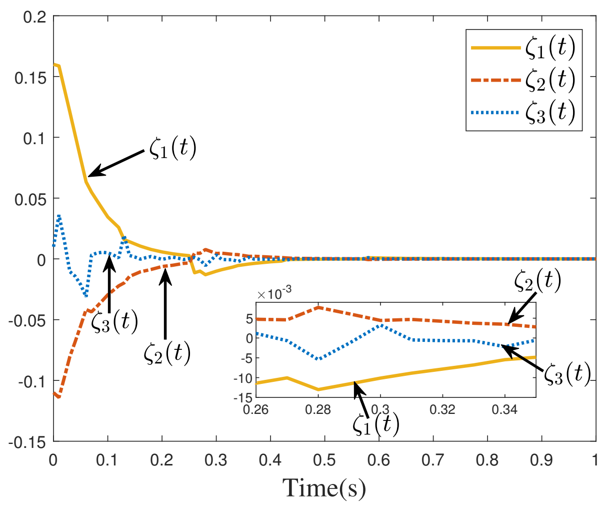

Fig. 6. State trajectories.

$h_i(\cdot) = tanh(\cdot)$, $i \in \{1, 2, 3\}$.

Assume that the initial state of the NN is $\zeta(0) = [0.16, -0.11, 0.01]^T$. When there is no control input, the chaotic behavior of the NN is depicted in Fig. 2.

We choose $\xi = 1.5$, $\varrho_1 = 0.1$, $\varrho_2 = 0.1$, $J = diag\{1, 1, 1\}$, the probability of replay attacks occurring $\alpha(t) = 0.1$, the sample interval $h = 0.01$, and the trigger threshold $\sigma = 0.1$. According to Theorem 2, the controller gain and the event trigger matrix can be obtained as follows:

$$K = \begin{bmatrix} -10.7984 & 0.9611 & 1.3261 \\ 0.5871 & -11.6407 & 17.7791 \\ 0.6071 & 13.1220 & -95.2104 \end{bmatrix},$$

$$\Phi = 10^{-8} \times \begin{bmatrix} 0.0832 & -0.0012 & 0.0023 \\ -0.0012 & 0.1041 & 0.0166 \\ 0.0023 & 0.0166 & 0.0697 \end{bmatrix}.$$

The trajectory of the Bernoulli variable $\alpha(t)$ is presented in Fig. 3, the triggering instants and triggering intervals of the ETM are described in Fig. 4, the evolution of the control inputs is illustrated in Fig. 5, and the trajectories of the state of the CLS are shown in Fig. 6. It can be seen that under the given controller gain and event trigger matrix, the system state quickly converges to 0, thereby proving the effectiveness of the design method.

V. Conclusion

This paper investigated the event-triggered stabilization problem for NNs under replay attacks. By introducing an ETM (2) and controller (4), NN (1) was redefined as a switched CLS (5). Subsequently, a criterion for the MSE stability of CLS was established in Theorem 1. Based on Theorem 1, a co-design method for event trigger matrix Φ and controller gain K were proposed in Theorem 2. Finally, a Hopfield NN example was given to illustrate the effectiveness of the proposed method.

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